

Generalized Hasse invariants for Shimura varieties of Hodge type

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ABSTRACT. We construct canonical Hasse invariants for arbitrary Shimura varieties of Hodge type.

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Introduction

The Hasse invariant is a modular form which has been an important tool for constructing congruences between automorphic forms. It is defined for certain Shimura varieties that are endowed with a universal abelian scheme (like Siegel Shimura varieties, Hilbert-Blumenthal varieties, or more generally Shimura varieties of Hodge type). Its non-vanishing locus on the special fiber in positive characteristic p is the locus where the universal abelian scheme is ordinary. It is a section of the $(p-1)$ -th power of the Hodge line bundle of the universal abelian scheme.

In this paper we consider the following situation. Let (\mathbf{G}, \mathbf{X}) be a Shimura datum of Hodge type, $K = K_p K^p \subseteq \mathbf{G}(\mathbb{A}_f)$ an open compact subgroup such that $K_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ is hyperspecial and such that $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ is sufficiently small. Let $S = S_K(\mathbf{G}, \mathbf{X})$ be the special fiber of the canonical integral model of a Shimura variety of Hodge type at a place v of the reflex field E lying over p . Then usually the ordinary locus is empty, hence the classical Hasse invariant vanishes (in fact, in the PEL case the ordinary locus is non-empty if and only if $E_v = \mathbb{Q}_p$ by [Wd1] Theorem 1.6.3). Instead, the generic Newton stratum for good reductions of Shimura varieties of Hodge type is the so-called μ -ordinary locus S^μ , introduced by the second author in [Wd1], which has been shown to be open and dense by Wortmann ([Wor]). In this paper we define an integer $N \geq 1$, the Hasse number (Definition 4.11). It depends only on the special fiber G of the reductive group scheme over \mathbb{Z}_p defined by K_p and on the conjugacy class of cocharacters defining the reflex field. Then our main result is the following.

Theorem 1. *There exists a canonical section H of the N -th power of the Hodge line bundle ω on S such that the non-vanishing locus of H is S^μ .*

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We refer to the main text (Theorem 4.12) for the precise meaning of “canonical”. If \mathbf{G} is split over \mathbb{Q}_p and its derived group is simply connected, then the ordinary locus is non-empty, we obtain $N = p - 1$ (Corollary 2.21) and H is the classical Hasse invariant.

All other generalizations of Hasse invariants in positive characteristic that we are aware of are special cases of our construction. We give two examples: For Hilbert-Blumenthal varieties partial Hasse invariants have been constructed by Goren in [Gor]. This is rephrased in our language in Example 3.19. For Shimura varieties of unitary PEL type, Hasse invariants have been considered by Goldring and Nicole in [GoNi1] and [GoNi2]. We consider a special case in Example 2.23 if $\mathbf{G}_{\mathbb{Q}_p}$ is non-split; here the Hasse number is $p^2 - 1$.

A formal argument using the definition of the minimal compactification and its normality shows that the Hasse invariant H can be extended to the minimal compactification and that some power of H can be lifted to characteristic 0. We define the μ -ordinary locus $S^{\min, \mu}$ of the minimal compactification S^{\min} as the non-vanishing locus of the extension of H . As the Hodge line bundle on the minimal compactification is ample, we deduce the following corollary.

Corollary 2. *The μ -ordinary locus $S^{\min, \mu}$ of the minimal compactification is affine.*

The idea behind our construction is to consider the Ekedahl-Oort stratification of S which is given by a smooth morphism $\zeta: S \rightarrow G\text{-Zip}^\chi$ constructed by Zhang ([Zha1], see also [Wor]). Here G is a reductive reduction of \mathbf{G} over \mathbb{F}_p , χ is a certain cocharacter of G , defined over a finite extension of \mathbb{F}_p , given by the Shimura datum and $G\text{-Zip}^\chi$ is the algebraic stack of G -zips of type χ introduced by Pink, Ziegler, and the second author in [PWZ2] (see Subsection 4.4 for details). By a result of Wortmann ([Wor]), the inverse image of the generic point of $G\text{-Zip}^\chi$ is the μ -ordinary locus S^μ .

We then construct a line bundle ω^b on $G\text{-Zip}^\chi$ whose pull back via ζ is the Hodge line bundle on S . Moreover we show that the space of global sections of any power $(\omega^b)^{\otimes k}$ over $G\text{-Zip}^\chi$ has dimension at most 1 and that there is a non-vanishing section H^b for $k = N$. Then our main technical result (Theorem 3.8) shows that its vanishing locus is precisely the complement of the generic point of $G\text{-Zip}^\chi$. Hence $H := \zeta^*(H^b)$ is the desired Hasse invariant.

We now give an overview of the paper. As line bundles on quotient stacks (like $G\text{-Zip}^\chi$) are just line bundles with equivariant structures, we recall in Section 1 several results on equivariant Picard groups. None of them are probably new but for many of them we did not find a reference or only references with too restrictive hypotheses (such as that the ground field is of characteristic 0). Thus we included also some proofs.

In Section 2 we examine the Picard group of the stack of G -zips and of related stacks like the Bruhat stack. We then state in Section 3 a positivity conjecture and prove the conjecture in a special case which is sufficient for the applications to Shimura varieties stated above. This is the technical heart of our paper.

In the last section we deduce the above applications to Shimura varieties of Hodge type.

Notation and terminology: A linear algebraic group G over a field k is an affine smooth group scheme over k . We denote by $X^*(G)$ its group of k -rational characters.

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1 Equivariant Picard groups

In this section, G denotes an arbitrary connected linear algebraic group over an algebraically closed field k . Products of schemes are fiber products over k . A variety is an integral k -scheme of finite type. A G -scheme is a k -scheme endowed with a G -action $G \times X \rightarrow X$.

We always consider left actions. If G is an algebraic group and H is an algebraic subgroup we denote by G/H the quotient of G by the left action $(h, g) \mapsto gh^{-1}$.

1.1 G -linearization of a line bundle

Let X be a G -scheme, where the action is given by a map $a: G \times X \rightarrow X$. Let \mathcal{L} be a line bundle on X . Define the projections $p_{23}: G \times G \times X \rightarrow G \times X$, $(g, h, x) \mapsto (h, x)$ and $p_2: G \times X \rightarrow X$, $(g, x) \mapsto x$. Finally, write μ_G for the multiplication map $G \times G \rightarrow G$.

Definition 1.1. A G -linearization of \mathcal{L} is an isomorphism $\phi: a^*(\mathcal{L}) \rightarrow p_2^*(\mathcal{L})$ satisfying the cocycle condition

$$p_{23}^*(\phi) \circ (\text{id}_G \times a)^*(\phi) = (\mu_G \times \text{id}_X)^*(\phi)$$

If $\pi: L \rightarrow X$ is the corresponding geometric line bundle, a linearization of L is an action $a_L: G \times L \rightarrow L$ such that :

- The map $\pi: L \rightarrow X$ is G -invariant.
- The induced isomorphism $G \times L \rightarrow a^*(L)$ is a morphism of geometric line bundles.

The first condition ensures that we have a cartesian diagram :

$$\begin{array}{ccc} G \times L & \xrightarrow{a_L} & L \\ \text{id}_G \times \pi \downarrow & & \downarrow \pi \\ G \times X & \xrightarrow{a} & X \end{array}$$

The second condition says that $G \times L \simeq a^*(L)$ as geometric line bundles. We denote by $\text{Pic}^G(X)$ the group of isomorphism classes of G -linearized line bundles on X . There is a natural map

$$\text{Pic}^G(X) \rightarrow \text{Pic}(X)$$

whose image is the subgroup of G -linearizable line bundles, denoted by $\text{Pic}_G(X)$.

The group $\text{Pic}^G(X)$ can be identified with the Picard group of the quotient stack $[G \backslash X]$. Then $\text{Pic}^G(X) \rightarrow \text{Pic}(X)$ is the homomorphism given by pull back by the projection $X \rightarrow [G \backslash X]$. Thus, if $H \subset G$ are linear algebraic groups, there is an isomorphism $\text{Pic}(G/H) \simeq \text{Pic}^H(G)$. Notice that in this case, $[G/H] = G/H$ is a scheme.

1.2 General results on equivariant Picard groups

Let us first recall the following result by Rosenlicht ([FoIv] Corollary 2.2).

Theorem 1.2 (Rosenlicht). *Let G be a connected linear algebraic group and let $f: G \rightarrow \mathbb{G}_m$ be a morphism of k -schemes such that $f(1) = 1$. Then f is a character.*

For every k -scheme X , we define :

$$\mathcal{E}(X) := \mathbb{G}_m(X)/k^\times.$$

where $\mathbb{G}_m(X) = \mathcal{O}(X)^\times$. Recall the following result from [KKV] 1.3.

Lemma 1.3. *If X is an integral k -scheme of finite type, then $\mathcal{E}(X)$ is a finitely generated free abelian group.*

This implies that if X carries an algebraic action by an algebraic group G , then the induced action of G on $\mathcal{E}(X)$ factors over $\pi_0(G)$, the finite group of components of G . Note that if H is an algebraic group, then the inclusion $X^*(H) \subset \mathbb{G}_m(H)$ induces an isomorphism

$$X^*(H) \cong \mathcal{E}(H)$$

by Rosenlicht's theorem.

In the statement of the next proposition we denote by $H^1(\ , \)$ group cohomology and by $H_{\text{alg}}^1(G, \mathbb{G}_m(X))$ the subgroup of classes of algebraic cocycles, i.e., of maps $c: G(k) \rightarrow \mathbb{G}_m(X)$ corresponding to a morphism $G \times X \rightarrow \mathbb{G}_m$.

Proposition 1.4. *Let G be a smooth algebraic group, and let X be an irreducible G -variety. Then there are exact sequences:*

$$0 \rightarrow \frac{\mathbb{G}_m(X)^G}{k^\times} \rightarrow \mathcal{E}(X)^G \rightarrow X^*(G) \rightarrow H_{\text{alg}}^1(G, \mathbb{G}_m(X)) \rightarrow H^1(\pi_0(G)(k), \mathcal{E}(X))$$

$$0 \rightarrow H_{\text{alg}}^1(G, \mathbb{G}_m(X)) \rightarrow \text{Pic}^G(X) \rightarrow \text{Pic}(X)$$

If X is normal and G connected, the second exact sequence has an extension by a map $\text{Pic}(X) \rightarrow \text{Pic}(G)$.

Proof. This is [KKV], Proposition 2.3 and Lemma 2.2. The general assumption that k is of characteristic 0 is not needed in the proof there, only the smoothness of the algebraic group G is used. \square

If G is connected, the two exact sequences of Proposition 1.4 combine into a longer one. We obtain the following result.

Theorem 1.5. *Let G be a smooth connected algebraic group and let X be a normal irreducible G -variety over k . Then we have an exact sequence of abelian groups*

$$1 \rightarrow k^\times \rightarrow \mathbb{G}_m(X)^G \rightarrow \mathcal{E}(X) \rightarrow X^*(G) \rightarrow \text{Pic}^G(X) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(G)$$

For later use let us give the explicit construction of these maps.

- If $f \in \mathbb{G}_m(X)$, then for any $g \in G$, there is $\lambda_f(g) \in k^\times$ such that

$$g \cdot f = \lambda_f(g)f.$$

Then λ_f is a character of G . This defines the map $\mathcal{E}(X) \rightarrow X^*(G)$, $f \mapsto \lambda_f$.

- Then, if $\lambda \in X^*(G)$, we define a G -linearization of the trivial line bundle on X

$$G \times X \times \mathbb{A}_k^1 \rightarrow X \times \mathbb{A}_k^1$$

by $(g, x, s) \mapsto (g \cdot x, \lambda(g)s)$. This yields a map $X^*(G) \rightarrow \text{Pic}^G(X)$.

- Finally, the map $\text{Pic}(X) \rightarrow \text{Pic}(G)$ which terminates the sequence depends on the choice of an element $x_0 \in X$. An element $L \in \text{Pic}(X)$ is sent to $a^*(L)|_{G \times \{x_0\}}$.

Corollary 1.6. *Let G be a smooth connected algebraic group, let H be a smooth connected algebraic subgroup of G , and let $\pi: G \rightarrow G/H$ be the quotient by left action $(h, g) \mapsto gh^{-1}$. Then there is an exact sequence*

$$1 \rightarrow X^*(G)^H \rightarrow X^*(G) \rightarrow X^*(H) \xrightarrow{\xi} \text{Pic}(G/H) \rightarrow \text{Pic}(G) \rightarrow \text{Pic}(H).$$

The homomorphism ξ attaches to $\lambda \in X^*(H)$ the invertible $\mathcal{O}_{G/H}$ -module \mathcal{L}_λ whose sections over $U \subseteq G/H$ are given by

$$(1.1) \quad \Gamma(U, \mathcal{L}_\lambda) = \{ f: \pi^{-1}(U) \rightarrow k; f(gh^{-1}) = \lambda(h)f(g) \text{ for } g \in \pi^{-1}(U), h \in H \}.$$

The corresponding geometric bundle is $(G \times \mathbb{A}_k^1)/H$, where H acts on \mathbb{A}_k^1 via $\lambda: H \rightarrow \mathbb{G}_m$.

Remark 1.7. If $\pi: G \rightarrow G/H$ has locally for the Zariski topology a section, then $i^*: \text{Pic}(G) \rightarrow \text{Pic}(H)$ is surjective ([San] Prop. 6.10).

If the group acting is not connected, we do not have in general a long exact sequence as in Theorem 1.5. However, this is true in the case of G/H where $H \subset G$ is a subgroup (or more generally when the group acts trivially on $\mathcal{E}(X)$).

Proposition 1.8. *Let G be a smooth connected algebraic group and H a smooth subgroup of G . Then there is an exact sequence:*

$$0 \rightarrow \mathcal{E}(G/H) \rightarrow X^*(G) \rightarrow X^*(H) \rightarrow \text{Pic}^H(G) \rightarrow \text{Pic}(G).$$

Proof. We apply Proposition 1.4 for $X = G$ and $G = H$. The group H acts trivially on $\mathcal{E}(G) = X^*(G)$, since $\chi(gh^{-1}) = \chi(g)\chi(h)^{-1}$ for all $g \in G, h \in H, \chi \in X^*(G)$, so $h \cdot \chi = \chi$ in $\mathcal{E}(G)$. It follows that $H^1(\pi_0(H), \mathcal{E}(G)) = \text{Hom}_{\text{Grp}}(\pi_0(H), \mathcal{E}(G)) = 0$ because $\pi_0(H)$ is finite and $\mathcal{E}(G)$ is torsion-free. \square

We will also need the following proposition.

Proposition 1.9. *Let $H \subset G$ be algebraic groups. Then there is a natural isomorphism:*

$$\text{Pic}^G(G/H) \simeq X^*(H).$$

Proof. One has $\text{Pic}^G(G/H) \simeq \text{Pic}([G \backslash (G/H)]) \simeq \text{Pic}([1/H]) \simeq \text{Pic}^H(1) \simeq X^*(H)$. \square

1.3 Functoriality of the equivariant Picard group

Let G, G' be algebraic groups and $f: G \rightarrow G'$ a morphism of algebraic groups. Let X be a G -scheme, let X' be a G' -scheme, and $\pi: X \rightarrow X'$ be an f -equivariant morphism, i.e. a morphism of k -schemes such that

$$(1.2) \quad \pi(g \cdot x) = f(g) \cdot \pi(x) \quad \forall g \in G, x \in X$$

Then π induces naturally a homomorphism

$$(1.3) \quad \pi^*: \text{Pic}^{G'}(X') \rightarrow \text{Pic}^G(X)$$

by interpreting π^* as the pull-back by the induced map of stacks

$$[G \backslash X] \rightarrow [G' \backslash X'] .$$

A more concrete description of π^* is the following. Let $L' \in \text{Pic}^{G'}(X')$ be a (geometric) G' -linearized line bundle. Then define a G -action on $L = X \times_{X'} L'$ by

$$(g, (x, l')) \mapsto (gx, (gx, f(g)l')) .$$

If X and X' are smooth varieties and if G, G' are smooth and connected, we obtain a natural morphism between the exact sequences provided by Theorem 1.5.

1.4 The case of algebraic groups defined over finite fields

In this section we assume that k is an algebraic closure of a finite field \mathbb{F}_q with q elements. We denote by $\sigma \in \text{Gal}(k/\mathbb{F}_q)$ the arithmetic Frobenius $a \mapsto a^q$. We begin with the well-known theorem of Lang (e.g., [St]):

Theorem 1.10 (Lang). *Let G be a connected algebraic group defined over \mathbb{F}_q . Let $\varphi: G \rightarrow G$ be the q -th power Frobenius. Then the Lang map*

$$L: G \rightarrow G, \quad x \mapsto \varphi(x)x^{-1}$$

is finite étale and surjective.

Corollary 1.11. *Let G be a connected algebraic group defined over \mathbb{F}_q . Then the Lang map L of Theorem 1.10 induces an isomorphism of varieties $G/G(\mathbb{F}_q) \xrightarrow{\sim} G$.*

Proof. Clearly L induces an injective map $G/G(\mathbb{F}_q) \rightarrow G$ which is surjective and finite étale by Theorem 1.10 and hence an isomorphism. \square

Proposition 1.12. *Let G be a connected linear algebraic group defined over \mathbb{F}_q . Assume that $\text{Pic}(G) = 0$. Then there is an exact sequence :*

$$0 \rightarrow X^*(G) \rightarrow X^*(G) \rightarrow \text{Hom}(G(\mathbb{F}_q), k^\times) \rightarrow 0$$

where the first map is $\chi \mapsto \sigma \cdot \chi - \chi$.

Proof. If $\sigma \cdot \chi = \chi$, the image of χ is contained in the finite group $\mu_{q-1}(k)$. As G is connected, this implies that χ is trivial. Hence the first map is injective. The sequence is clearly exact in the middle. Write $H = G(\mathbb{F}_q)$. The Lang map $L: G \rightarrow G, x \mapsto \varphi(x)x^{-1}$ induces an isomorphism of varieties $G/H \rightarrow G$, so we deduce that $\text{Pic}^H(G) = \text{Pic}(G/H) = 0$. Hence $X^*(G) \rightarrow X^*(H)$ is surjective by Proposition 1.8. \square

Remark 1.13. This result is false without the condition $\text{Pic}(G) = 0$. Consider $G = \text{PGL}_n$ over \mathbb{F}_q . Then $G(\mathbb{F}_q) = \text{GL}_n(\mathbb{F}_q)/\mathbb{F}_q^\times$ because $H^1(\text{Gal}(k/\mathbb{F}_q), k^\times) = 0$ by Hilbert 90. Then the derived group of $G(\mathbb{F}_q)$ is $\text{SL}_n(\mathbb{F}_q)/\mu_n(\mathbb{F}_q)$ and the abelianization of $G(\mathbb{F}_q)$ is $\mathbb{F}_q^\times/\mathbb{F}_q^{\times n} \simeq \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(q-1, n)$. It follows that $\text{Hom}(G(\mathbb{F}_q), k^\times) \simeq \mathbb{Z}/d\mathbb{Z}$, whereas $X^*(G) = 0$.

Note that $\text{Pic}(G) \cong \mathbb{Z}/n\mathbb{Z}$ by Proposition 2.1 below.

Example 1.14. For example, if $G = \text{SL}_n$, then the above proposition implies that there are no nontrivial group homomorphisms $\text{SL}_n(\mathbb{F}_q) \rightarrow k^\times$.

1.5 G -varieties and divisors

Let X be a k -variety. As usual we denote the free group generated by the irreducible subvarieties of codimension $i \geq 0$ by $Z^i(X)$. Elements of $Z^1(X)$ are the Weil divisors on X .

If X is a G -variety, we define an action of G on $Z^i(X)$ in the obvious way: For $D = \sum n_C [C] \in Z^i(X)$ we set

$$g \cdot D = \sum n_C [g \cdot C].$$

Let $Z^i(X)^G$ be the subgroup of G -invariant elements of $Z^i(X)$.

Assume that X is regular. Recall that the locally free \mathcal{O}_X -module $\mathcal{L}(D)$ of rank 1 associated to $D = \sum n_C [C] \in Z^1(X)$ is defined by

$$\mathcal{L}(D)(U) = \{ f \in K(X) ; v_C(f) + n_C \geq 0 \text{ for all } C \text{ intersecting } U \}.$$

Thus we see that when $D \in Z^1(X)^G$ and $g \in G$, there is a natural isomorphism $a_g: g^* \mathcal{L}(D) \rightarrow \mathcal{L}(D)$ defined by $f \mapsto g \cdot f$ (notice that $\text{div}(g \cdot f) = g \cdot \text{div}(f)$). These isomorphisms $(a_g)_{g \in G}$ satisfy the cocycle condition $a_{gh} = a_h \circ (h^* a_g)$ so in turn, they define a G -linearization of $\mathcal{L}(D)$. We have constructed a commutative diagram

$$\begin{array}{ccc} Z^1(X)^G & \hookrightarrow & Z^1(X) \\ \downarrow \mathcal{L} & & \downarrow \\ \text{Pic}^G(X) & \longrightarrow & \text{Pic}(X) \end{array}$$

Of course this is a very special instance of the theory of equivariant Chow groups. Assume from now on that X is a smooth G -variety.

Proposition 1.15. *Let G be a smooth algebraic group and let X be an irreducible*

smooth G -variety. There is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & & H_{\text{alg}}^1(G, \mathbb{G}_m(X)) \hookrightarrow & H^1(G, \mathbb{G}_m(X)) & \\
& & & & \downarrow & \downarrow & \\
0 & \longrightarrow & \frac{K(X)^{\times G}}{\mathbb{G}_m(X)^G} & \longrightarrow & Z^1(X)^G & \xrightarrow{\mathcal{L}} & \text{Pic}^G(X) \longrightarrow H^1(G, K(X)^{\times}) \\
& & \downarrow & & \downarrow = & & \downarrow \\
0 & \longrightarrow & \left(\frac{K(X)^{\times}}{\mathbb{G}_m(X)} \right)^G & \longrightarrow & Z^1(X)^G & \longrightarrow & \text{Pic}_G(X) \longrightarrow H^1\left(G, \frac{K(X)^{\times}}{\mathbb{G}_m(X)}\right) \\
& & \downarrow & & & & \downarrow \\
& & H_{\text{alg}}^1(G, \mathbb{G}_m(X)) & & & & 0 \\
& & \downarrow & & & & \\
& & H^1(G, K(X)^{\times}) & & & &
\end{array}$$

Proof. For us only the central square of this diagram will be important. First, let us define the maps. Let $\mathcal{L} \in \text{Pic}_G(X)$ be a G -linearizable line bundle on X . We may assume that $\mathcal{L} = \mathcal{L}(D)$ for some Weil divisor D on X . Then $g \cdot D = D + \text{div}(f_g)$ for some $f_g \in \frac{K(X)^{\times}}{\mathbb{G}_m(X)}$. This defines a cocycle, independant of the choice of D . We get a map $\text{Pic}_G(X) \rightarrow H^1\left(G, \frac{K(X)^{\times}}{\mathbb{G}_m(X)}\right)$.

Now, let $\tilde{\mathcal{L}} \in \text{Pic}^G(X)$ and let $\mathcal{L} \in \text{Pic}(X)$ be the underlying line bundle. Again, we may assume that $\mathcal{L} = \mathcal{L}(D)$ for some Weyl divisor D . The G -linearization of \mathcal{L} gives an isomorphism $\mathcal{L}(gU) \rightarrow \mathcal{L}(U)$ for all open subsets U and all $g \in G$. Choose $U \subset X - |D|$. Then $\mathcal{L}(U) = \mathcal{O}_X(U)$, so the element $1 \in \mathcal{L}(U)$ is mapped to some $\lambda_g \in \mathcal{L}(gU) \subset K(X)$. This defines a 1-cocycle $g \mapsto \lambda_g$ in $H^1(G, K(X)^{\times})$.

Finally, we define also in the above diagram the two maps with target $Z^1(X)^G$ by $f \mapsto \text{div}(f)$. The commutativity of the diagram is an easy exercise.

First, we prove the exactness of the upper horizontal sequence at the term $Z^1(X)^G$. So let D be a G -invariant Weil divisor, mapping to zero in $\text{Pic}^G(X)$. In particular, we can write $D = \text{div}(f)$ with some $f \in K(X)^{\times}$. Then the map

$$\gamma : g \mapsto \frac{g \cdot f}{f}$$

is an algebraic cocycle mapping to zero in $\text{Pic}^G(X)$. It follows that this cocycle is a coboundary, so there is a function $h \in \mathbb{G}_m(X)$ such that $\gamma(g) = \frac{g \cdot h}{h}$ for all $g \in G$. It follows that $\frac{f}{h} \in K(X)^{\times G}$ and this shows the first part.

Secondly, we prove exactness of the lower horizontal sequence at the term $\text{Pic}_G(X)$. Let $\mathcal{L} = \mathcal{L}(D)$ be a G -linearizable line bundle on X , such that the associated cocycle

of $\frac{K(X)^\times}{\mathbb{G}_m(X)}$ is a coboundary. By definition, there is $f \in K(X)^\times$ such that $g \cdot D = D + \operatorname{div}\left(\frac{g \cdot f}{f}\right)$ and it follows that $D - \operatorname{div}(f)$ is a G -invariant Weil divisor mapping to \mathcal{L} . We will not need the rest of the diagram, so we leave the proofs to the reader. \square

1.6 G -varieties with open orbit

In this subsection we assume that X is a smooth irreducible G -variety which has an open G -orbit $U \subset X$. In this case it is clear that $K(X)^G = k$, so the map $\mathcal{L}: Z^1(X)^G \rightarrow \operatorname{Pic}^G(X)$ is injective by Proposition 1.15. We assume further that G is connected. Then G necessarily acts on the finitely many irreducible components of $X \setminus U$ trivially. Therefore the group $Z^1(X)^G$ is the free group generated by those irreducible components in $X \setminus U$ which are of codimension 1 in X .

Proposition 1.16. *There is the following commutative diagram with exact rows and columns.*

$$(1.4) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \frac{\mathcal{E}(U)}{\mathcal{E}(X)} & & \\ & & & & \downarrow & & \\ & & & 0 & Z^1(X)^G & = & Z^1(X)^G \\ & & & \downarrow & \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ 0 & \longrightarrow & \mathcal{E}(X) & \longrightarrow & X^*(G) & \longrightarrow & \operatorname{Pic}^G(X) \longrightarrow \operatorname{Pic}_G(X) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}(U) & \longrightarrow & X^*(G) & \longrightarrow & \operatorname{Pic}^G(U) \longrightarrow \operatorname{Pic}_G(U) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Proof. The commutativity is clear. The exactness of the two rows is a simple application of Theorem 1.5 using the fact that $\mathcal{O}(X)^G = \mathcal{O}(U)^G = k$. Let us prove the surjectivity of the last two vertical maps. Note that we only need to prove the surjectivity of $\operatorname{Pic}^G(X) \rightarrow \operatorname{Pic}^G(U)$. From Theorem 1.5 and functoriality we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}(X) & \longrightarrow & X^*(G) & \longrightarrow & \operatorname{Pic}^G(X) & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & \operatorname{Pic}(G) \\ & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{E}(U) & \longrightarrow & X^*(G) & \longrightarrow & \operatorname{Pic}^G(U) & \longrightarrow & \operatorname{Pic}(U) & \longrightarrow & \operatorname{Pic}(G) \end{array}$$

where the rightmost horizontal maps have to be defined using the same choice of an element in U . As X is smooth, $\text{Pic}(X) \rightarrow \text{Pic}(U)$ is surjective and a simple diagram chase shows the surjectivity of $\text{Pic}^G(X) \rightarrow \text{Pic}^G(U)$.

An element in the kernel of $\text{Pic}_G(X) \rightarrow \text{Pic}_G(U)$ comes from a Weil divisor D with support in $X \setminus U$, which is G -stable since $Z^1(X)^G = Z_{n-1}(X - U)$. This proves the exactness of the rightmost column. It remains to show that the kernel of $\text{Pic}^G(X) \rightarrow \text{Pic}^G(U)$ is $Z^1(X)^G$. Let $\tilde{\mathcal{L}} \in \text{Pic}^G(X)$ be mapped to zero in $\text{Pic}^G(U)$. Since we proved the exactness of the rightmost column, we may assume further that the underlying line bundle \mathcal{L} is the trivial on X , so it comes from a character of G . Since it is trivial in $\text{Pic}^G(U)$, this character is attached to a function $f \in \mathcal{E}(U)$. Then the opposite of the divisor of f is a preimage of $\tilde{\mathcal{L}}$ in $Z^1(X)^G$. \square

Remark 1.17. From the above diagram, we get a natural map $\mathcal{E}(U) \rightarrow X^*(G) \rightarrow \text{Pic}^G(X)$, whose image is contained in $Z^1(X)^G$. As we mentioned in the proof, this map is $f \mapsto -\text{div}(f)$.

The choice of a point $x_0 \in U$ yields an isomorphism $G/G_{x_0} \cong U$, where G_{x_0} is the (scheme-theoretic) stabilizer of x_0 . If G_{x_0} is finite (equivalently, $\dim(G) = \dim(X)$), the orbit map

$$u: G \mapsto U, \quad g \mapsto g \cdot x_0$$

is a finite flat G -equivariant morphism whose degree is $N := \dim_k \mathcal{O}(G_{x_0})$. It is finite étale if and only if G_{x_0} is reduced.

The morphism u induces a group homomorphism $\mathcal{E}(U) \rightarrow X^*(G)$ whose image is the subgroup of $X^*(G)$ of characters which vanish on G_{x_0} . The natural isomorphism $X^*(G_{x_0}) \simeq \text{Pic}^G(U)$ from Proposition 1.9 induces an isomorphism of exact sequences

$$(1.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}(U) & \longrightarrow & X^*(G) & \longrightarrow & X^*(G_{x_0}) \longrightarrow \frac{X^*(G_{x_0})}{\text{im}(X^*(G) \rightarrow X^*(G_{x_0}))} \longrightarrow 0 \\ & & \downarrow = & & \downarrow = & & \downarrow \simeq \\ 0 & \longrightarrow & \mathcal{E}(U) & \longrightarrow & X^*(G) & \longrightarrow & \text{Pic}^G(U) \longrightarrow \text{Pic}_G(U) \longrightarrow 0 \end{array}$$

For example, assume that G is defined over \mathbb{F}_q , acting on itself via Frobenius-conjugation. Then $U = G$ and one has $\text{Pic}^G(U) \simeq \text{Hom}(G(\mathbb{F}_q), k^\times)$.

1.7 The space of global sections

Proposition 1.18. *Let G be an algebraic group and let X be an irreducible G -variety with an open orbit $U \subset X$. Denote by $\pi: X \rightarrow [X/G]$ the projection. Let \mathcal{L} be a line bundle on the stack $[X/G]$ and write $L = \pi^* \mathcal{L}$.*

- (1) *$H^0([X/G], \mathcal{L})$ identifies with $H^0(X, L)^G$.*
- (2) *The k -vector space $H^0([X/G], \mathcal{L})$ has dimension less than 1.*
- (3) *If $H^0([X/G], \mathcal{L}) \neq 0$ then \mathcal{L} restricts to the trivial line bundle on $[U/G]$.*
- (4) *If \mathcal{L} is trivial, $H^0([X/G], \mathcal{L}) = k$.*

Proof. A global section $s \in H^0([X/G], \mathcal{L})$ is the same as a morphism of $\mathcal{O}_{[X/G]}$ -modules $\mathcal{O}_{[X/G]} \rightarrow \mathcal{L}$, and this is the same as a morphism of \mathcal{O}_X -modules $\mathcal{O}_X \rightarrow L$ which commutes with the G -action (with respect to the trivial action on \mathcal{O}_X). This proves the first assertion.

The map $H^0(X, L) \rightarrow H^0(U, L)$ is injective and G -equivariant, so in order to prove the rest of the proposition, we may assume $X = U$ and show that only the trivial line bundle has global sections. Now let $H \subset G$ be the stabilizer of some element in X . Then X identifies with G/H . Now, one has $\text{Pic}^G(G/H) \simeq X^*(H)$. If $\chi \in X^*(H)$ is a character, we denote by $\mathcal{L}(\chi)$ and $L(\chi)$ the corresponding line bundles on $[G \backslash G/H]$ and G/H respectively. The global sections of $L(\chi)$ on G/H is the set :

$$H^0(G/H, L(\chi)) \simeq \{f : G \rightarrow k, f(gh) = \chi(h)f(g), \forall g \in G, \forall h \in H\}$$

and the G -invariant global sections are zero unless $\chi = 1$. \square

Let $A \subset \text{Pic}([X/G])$ be the set of line bundles admitting global sections. We have seen that the set A is contained in

$$Z^1(X)^G = \text{Ker}(\text{Pic}([X/G]) \rightarrow \text{Pic}([U/G]))$$

Then A is equal to the cone of effective divisors in $Z^1(X)^G$. Indeed, if $C \in Z^1(X)^G$ and $\mathcal{L}(C) \in \text{Pic}^G(X)$ is the associated line bundle, then the global sections of $\mathcal{L}(C)$ are zero or the constant functions $k \subset K(X)$ if $C \geq 0$.

On the other hand, if $\chi \in X^*(G)$ and $\mathcal{L}(\chi) \in \text{Pic}^G(X)$ is the associated line bundle, then $\mathcal{L}(\chi) \in A$ if and only if there is a nonzero function $f : X \rightarrow k$ such that

$$f(g \cdot x) = \chi(g)f(x), \quad \forall g \in G, \forall x \in X$$

In this representation of $\mathcal{L}(\chi)$, the global sections of $\mathcal{L}(\chi)$ identify with the space (of dimension 1) of all such functions f . This difference in the interpretation of the space of global functions lies in the fact that we have two separate ways to define an element of $\text{Pic}([X/G])$.

2 The Picard group of the stack of G -zip

In this section we will compute the Picard group of the stack $[E \backslash G]$ which was studied in detail in [PWZ1] and [PWZ2]. All schemes and in particular all algebraic groups are schemes over an algebraically closed field k .

2.1 The Picard group of a flag variety

First recall some general facts about the Picard group of a linear group:

Proposition 2.1. *Let G be a connected linear algebraic group over k . Let $G^{\text{red}} = G/R_u(G)$ be its maximal reductive quotient, let G^{der} the derived group of G^{red} and let $\pi : \tilde{G} \rightarrow G^{\text{der}}$ be the simply connected cover of G^{der} . Then :*

$$\text{Pic}(G) = \text{Pic}(G^{\text{red}}) = \text{Pic}(G^{\text{der}}) = X^*(\text{Ker}(\pi)).$$

In particular $\text{Pic}(G)$ is a finite group.

Proof. This follows from the description of the Picard group in terms of root datum ([FoIv] Prop. 5.1). \square

Corollary 2.2. *Let G be a reductive group. Then the following assertions are equivalent.*

- (i) $\text{Pic}(G) = 0$.
- (ii) *The derived group G^{der} is simply connected.*
- (iii) *For any parabolic subgroup P and any Levi subgroup $L \subseteq P$ one has $\text{Pic}(L) = 0$.*

Proof. The equivalence of (i) and (ii) is clear by Proposition 2.1 (because $\text{Ker}(\pi)$ is diagonalizable). Moreover, if the derived group of G is simply connected, then this holds for every Levi subgroup L (e.g., [SGA3] Exp. XXI, 6.5.11). \square

For a linear algebraic group G and a parabolic subgroup $P \subset G$, we define the following integer:

$$(2.1) \quad m_P := \text{rk}_{\mathbb{Z}}(X^*(P)) - \text{rk}_{\mathbb{Z}}(X^*(G)).$$

Proposition 2.3. *Let G be a linear algebraic group over k and let $P \subset G$ be a parabolic subgroup. Then $\text{Pic}(G/P)$ is a free group of rank m_P .*

Proof. Replacing G by $G/R_u(G)$, we may assume G reductive. Let \tilde{G} be the simply connected cover of the derived group of G . Let $\phi: \tilde{G} \rightarrow G$ be the canonical homomorphism and let $\tilde{P} := \phi^{-1}(P)$. Then \tilde{P} is a parabolic subgroup of \tilde{G} and ϕ induces an isomorphism $\tilde{G}/\tilde{P} \simeq G/P$. As $\text{Pic}(\tilde{G}) = 0$ (Corollary 2.2) and $X^*(\tilde{G}) = 0$, Corollary 1.6 yields an isomorphism

$$X^*(\tilde{P}) \xrightarrow{\sim} \text{Pic}(\tilde{G}/\tilde{P}) = \text{Pic}(G/P).$$

so $\text{Pic}(G/P)$ is free. Finally, we have an exact sequence:

$$0 \rightarrow X^*(G) \rightarrow X^*(P) \rightarrow \text{Pic}(G/P) \rightarrow \text{Pic}(G),$$

which shows that $\text{Pic}(G/P)$ has rank m_P . \square

Now let G be a reductive group over k and let P be a parabolic subgroup of G . Fix a Borel pair (B, T) of G such that $T \subset B \subset P$. Let $(X, \Phi, X^\vee, \Phi^\vee, \Delta)$ be the corresponding based root datum. Denote by $W = W(G, T) := N_G(T)/T$ the Weyl group and by $I \subseteq W$ the set of simple reflections defined by B . For $\alpha \in \Phi$ we denote by $s_\alpha \in W$ the corresponding reflection. We obtain a bijection

$$(2.2) \quad \Delta \xrightarrow{\sim} I, \quad \alpha \mapsto s_\alpha.$$

There are natural bijections between the powerset of I and the set of parabolic subgroups of G containing B (these are called standard). If $J \subseteq I$ is a subset, the corresponding standard parabolic will be denoted P_J . Let M_J is the unique Levi subgroup of P_J containing T . Then we get an inclusion $W_J := W(M_J, T) \hookrightarrow W(G, T)$ such that (W_J, J) is a Coxeter system and

$$J = W_J \cap I.$$

Every parabolic subgroup Q of G is conjugate to a unique standard parabolic subgroup P_J and $J \subseteq I$ is called the *type* of P .

Proposition 2.4. *The integer m_P is equal to the cardinality of $I \setminus J$, where J is the type of P .*

Proof. Again, we may assume that G is simply connected. The set of fundamental weights corresponding to $I \setminus J$ is a basis of $X^*(P)_{\mathbb{Q}}$ and the result follows. \square

2.2 The Picard group of the Bruhat stack

In this section, we fix two parabolic subgroups P and Q of a linear algebraic group G . We consider the quotient stack $[P \backslash G / Q]$, which we call the Bruhat stack. It is the quotient stack associated to the action of $P \times Q$ on G defined by

$$(p, q) \cdot x = pxq^{-1}$$

Lemma 2.5. *Let X be a $G \times H$ -variety, where G, H are two linear algebraic groups. Assume that $\mathbb{G}_m(X)^G = \mathbb{G}_m(X)^H = k^\times$. Then there is an exact sequence*

$$(2.3) \quad 0 \rightarrow \mathcal{E}(X) \rightarrow \mathcal{E}(X) \oplus \mathcal{E}(X) \rightarrow \text{Pic}^{G \times H}(X) \rightarrow \text{Pic}^G(X) \oplus \text{Pic}^H(X).$$

If $\text{Pic}(X) = 0$, the last map is surjective. If $\text{Pic}^G(X)$ and $\text{Pic}^H(X)$ are free, then $\text{Pic}^{G \times H}(X)$ is also free.

Proof. The first map is the diagonal map. The second one is the composition of the natural maps

$$\mathcal{E}(X) \oplus \mathcal{E}(X) \rightarrow X^*(G) \oplus X^*(H) = X^*(G \times H) \rightarrow \text{Pic}^{G \times H}(X),$$

where the first homomorphism is injective by our hypothesis (Theorem 1.5). The third one is given by forgetting the action on one side. Then all but the last assertion follow from a diagram chasing in the following commutative diagram with exact lines

$$\begin{array}{ccccccc} \mathcal{E}(X) & \hookrightarrow & X^*(G \times H) & \longrightarrow & \text{Pic}^{G \times H}(X) & \longrightarrow & \text{Pic}_{G \times H}(X) \\ \downarrow & & \downarrow \simeq & & \downarrow & & \downarrow \\ \mathcal{E}(X) \oplus \mathcal{E}(X) & \hookrightarrow & X^*(G) \oplus X^*(H) & \longrightarrow & \text{Pic}^G(X) \oplus \text{Pic}^H(X) & \longrightarrow & \text{Pic}_G(X) \oplus \text{Pic}_H(X) \end{array}$$

The last assertion then follows from the exactness of (2.3) and because $\mathcal{E}(X)$ is a free abelian group (Lemma 1.3). \square

Remark 2.6. The image of $\text{Pic}^{G \times H}(X) \rightarrow \text{Pic}^G(X) \oplus \text{Pic}^H(X)$ consists of elements of the form (L, L) , where L is a line bundle on X endowed with a G -action and an H -action that commute with each other.

Proposition 2.7. *Let $P, Q \subset G$ be parabolic subgroups. Then $\text{Pic}^{P \times Q}(G)$ is free of rank $\text{rk}_{\mathbb{Z}}(X^*(P)) + \text{rk}_{\mathbb{Z}}(X^*(Q)) - \text{rk}_{\mathbb{Z}}(X^*(G))$.*

Proof. Theorem 1.5 gives an exact sequence

$$0 \rightarrow X^*(G)_{\mathbb{Q}} \rightarrow X^*(P \times Q)_{\mathbb{Q}} \rightarrow \text{Pic}^{P \times Q}(G)_{\mathbb{Q}} \rightarrow 0$$

and Lemma 2.5 shows that $\text{Pic}^{P \times Q}(G)$ is free, so we are done. \square

2.3 Frobenius zip datum

In the sequel we will be mainly consider the following situation: Let k be an algebraically closed extension of a finite field \mathbb{F}_q with $q = p^m$. Let G be a reductive group over k defined over \mathbb{F}_q . Denote by σ the q -th power Frobenius of k . For every k -scheme or morphism of k -schemes we denote its pullback under σ by $(\)^{(q)}$. We fix a Borel pair (B, T) of G defined over \mathbb{F}_q . Let B^- be the opposite Borel subgroup of B with respect to T . Let P be a parabolic of G defined over k containing B^- , let L be the unique Levi subgroup of P containing T , and let P^- be the opposite parabolic subgroup with respect to L . Define $Q = (P^-)^{(q)}$. Let $\varphi: G \rightarrow G = G^{(q)}$ be the relative q -th power Frobenius isogeny. We also denote its restriction $L \rightarrow M := L^{(q)}$ again by φ . Note that M is the unique Levi subgroup of Q containing T . Then (G, P, Q, φ) is an algebraic zip datum in the sense of [PWZ1] 10.1.

For $x \in P$, we denote by \bar{x} the image of x in $P/R_u(P) = L$, similarly for the image of $y \in Q$ in $Q/R_u(Q) = L^{(q)}$. The associated *zip group* is defined by

$$E := \{ (x, y) \in P \times Q ; \varphi(\bar{x}) = \bar{y} \}$$

and E acts on G by restricting the action of $P \times Q$ to E . Note that $\dim(E) = \dim(G)$. The quotient stack $[E \backslash G]$ is called the *stack of G -zips*. One has

$$E = \{ (u\ell, v\varphi(\ell)) ; u \in R_u(P), v \in R_u(Q), \ell \in L \}.$$

The subgroup $\tilde{E} = \{ (x, y) \in L \times M ; \varphi(x) = y \}$ is a Levi subgroup of E , isomorphic to L .

By [PWZ1] Proposition 7.3, E acts with finitely many orbits on the variety G . These orbits are parametrized as follows ([PWZ1] Theorem 7.5). Let (W, I) be the Weyl group of (G, T) with its set of simple reflections given by B . As B and T are defined over \mathbb{F}_q , the relative Frobenius $\varphi: G \rightarrow G^{(q)} = G$ induces an isomorphism $\bar{\varphi}$ of the Coxeter system (W, I) .

Let $J \subseteq I$ and $K \subseteq I$ be the type of P and Q , respectively. Then $K = \varphi(J)^{\text{opp}}$. For every $w \in W$ we choose a representative $\dot{w} \in \text{Norm}_G(T)$ such that $(w_1 w_2)^{\cdot} = \dot{w}_1 \dot{w}_2$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Let $w_{0,J} \in W_J$ and $w_0 \in W$ the longest elements. In particular $w_0^2 = w_{0,J}^2 = 1$.

Set $g_0 := (w_0 w_{0,J})^{\cdot}$. Hence $g_0 \in \text{Norm}_G(T)$ lifts $w_{0,J} w_0 \in W$. Then (B, T, g_0) is a frame in the sense of [PWZ1] Definition 3.6: As $(w_0 w_{0,J} w_0)^{\cdot} \in L \subseteq P$ and ${}^{w_0} B = B_-$, we have

$$\begin{aligned} B &= B^{(q)} \subseteq Q, & g_0 B &= {}^{w_0 w_{0,J} w_0} (B_-) \subseteq P, \\ \varphi(L \cap {}^{g_0} B) &= \varphi({}^{w_0 w_{0,J} w_0} (L \cap B_-)) = \varphi(L \cap B) = M \cap B, \\ \varphi({}^{g_0} T) &= T. \end{aligned}$$

By [PWZ1] Theorem 5.12 and Theorem 7.5 we obtain a bijection

$$(2.4) \quad {}^J W \xrightarrow{\sim} \{E\text{-orbits on } G\}, \quad w \mapsto O^w := E \cdot (g_0 \dot{w})$$

such that $\dim O^w = \ell(w) + \dim(P)$.

2.4 The Picard group of the stack of a Frobenius zip datum

Lemma 2.8. *One has $X^*(G)^E = 0$.*

Proof. Let χ be an E -invariant character of G . In particular, for all $x \in L$, one has $\chi(x) = \chi(\varphi(x))$, so $\chi = \chi \circ \varphi$ because the restriction $X^*(G) \rightarrow X^*(L)$ is injective. But χ is defined over some finite field, so for some $r \geq 1$, $\chi = \chi \circ \varphi^r = \sigma^r \circ \chi$. We conclude that the image of χ is finite, so $\chi = 1$. \square

Using that $\mathcal{E}(G) = X^*(G)$ by Rosenlicht's theorem we deduce from Theorem 1.5 an exact sequence

$$0 \longrightarrow X^*(G) \longrightarrow X^*(E) \longrightarrow \mathrm{Pic}^E(G) \longrightarrow \mathrm{Pic}_E(G) \longrightarrow 0.$$

Corollary 2.9. *One has $\dim_{\mathbb{Q}}(\mathrm{Pic}^E(G)_{\mathbb{Q}}) = m_P = \#(I \setminus J)$, where m_P was defined in (2.1) and where J is the type of P .*

Proof. As $E/R_u(E) \cong P/R_u(P) \cong L$, we have $X^*(E) \cong X^*(P)$. \square

Note that the inclusion $E \subseteq P \times Q$ yields a morphism of quotient stacks $[E \backslash G] \rightarrow [P \backslash G/Q]$ and thus a homomorphism $\beta : \mathrm{Pic}^{P \times Q}(G) \rightarrow \mathrm{Pic}^E(G)$. We have a morphism of exact sequences :

$$(2.5) \quad \begin{array}{ccccccc} X^*(G) & \hookrightarrow & X^*(P) \oplus X^*(Q) & \longrightarrow & \mathrm{Pic}^{P \times Q}(G) & \longrightarrow & \mathrm{Pic}_{P \times Q}(G) \longrightarrow 0 \\ \parallel & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ X^*(G) & \hookrightarrow & X^*(E) & \longrightarrow & \mathrm{Pic}^E(G) & \longrightarrow & \mathrm{Pic}_E(G) \longrightarrow 0. \end{array}$$

It can be checked immediately that this diagram induces isomorphisms $\mathrm{Ker}(\alpha) \cong \mathrm{Ker}(\beta)$ and $\mathrm{Coker}(\beta) \cong \mathrm{Coker}(\gamma)$. In particular, the cokernel of β is finite. If $\mathrm{Pic}(G) = 0$, the map β is surjective. In this case, any line bundle on the stack of G -zips arises as the pull-back of a line bundle on the Bruhat stack.

2.5 The open orbit and the stabilizer of $1 \in G$

Let $\eta = w_{0,J}w_0$ be the element of maximal length in JW . Then $O^\eta = E \cdot (g_0\eta) = E \cdot 1$ is the unique open orbit. The stabilizer of $1 \in G$ in E is a finite group scheme $S \subset E$ and $O^\eta \cong E/S$ is affine (in fact, every E -orbit in G is affine by [WdYa] Theorem 2.2). Hence the complement $G \setminus O^\eta$ is a Cartier divisor whose irreducible components are fixed by the E -action because E is connected. Hence (2.4) induces bijections

$$(2.6) \quad I \setminus J \leftrightarrow \{w \in {}^JW ; \ell(w) = \ell(\eta) - 1\} \leftrightarrow \{\text{irreducible components of } G \setminus O^\eta\},$$

where the first bijection is given by $s \mapsto s\eta$ and where the second bijection is $w \mapsto \overline{O^w}$. The remarks in the beginning of Subsection 1.6 show that $Z^1(G)^E$ is freely generated by the right hand side. Hence we obtain an identification $\mathbb{Z}^{I \setminus J} \cong Z^1(G)^E$. Moreover, by (1.4) and (1.5) we have an exact sequence

$$(2.7) \quad 0 \longrightarrow Z^1(G)^E \longrightarrow \mathrm{Pic}^E(G) \longrightarrow X^*(S) \longrightarrow 1.$$

We now study the stabilizer of 1. First, we recall a basic result on the intersection of two parabolic subgroups. If we write $g \in G$ we always work with R -valued points for a k -algebra R .

Proposition 2.10. *Let P and Q be two parabolics of a reductive group G with unipotent radicals U and V , respectively. Let T be a maximal torus contained in P and in Q and let $L \subset P$ and $M \subset Q$ be the Levi subgroups with respect to T .*

- (1) *The subgroups $P \cap Q$, $L \cap M$, $L \cap V$, $M \cap U$, $U \cap V$ are smooth and connected.*
- (2) *The group $(P \cap Q).U$ is a parabolic subgroup of G contained in P , with Levi subgroups $L \cap M$.*
- (3) *Any element $x \in P \cap Q$ can be written uniquely as a product $x = abcd$, with $a \in L \cap M$, $b \in L \cap V$, $c \in M \cap U$, $d \in U \cap V$.*

Proof. The smoothness of $P \cap Q$ follows from [SGA3] Exp. XXVI, Lemma 4.1.1. This implies the smoothness of the other subgroups. For the rest, see [DM] Proposition 2.1. \square

The last statement means that $P \cap Q$ is the product of the varieties $L \cap M$, $L \cap V$, $M \cap U$, $U \cap V$. In particular, we obtain the following result.

Corollary 2.11. *We keep the same notations as in Proposition 2.10. If $U \cap V = \{1\}$ and $M \cap U = \{1\}$, then $P \cap Q \subset L$.*

If $x \in P$, define $\theta_L^P(x) \in L$ to be the unique element of L such that there exists $u \in U$ satisfying $x = \theta_L^P(x).u$. We will use repeatedly the following corollary.

Corollary 2.12. *We keep the same notations as in Proposition 2.10.*

- (1) *For all $x \in P \cap Q$, one has $\theta_L^P(x) \in P \cap Q$.*
- (2) *For all $x \in P \cap Q$, one has $\theta_L^P(\theta_M^Q(x)) = \theta_M^Q(\theta_L^P(x))$ (note that this makes sense because of the first assertion).*
- (3) *Assume G is defined over \mathbb{F}_q and let $\varphi : G \rightarrow G$ the q -th power Frobenius. Then $\varphi(\theta_L^P(x)) = \theta_{\sigma_L^P}^{\sigma_L^P}(\varphi(x))$.*
- (4) *Assume $T \subset B \subset P \cap Q$ for some Borel B . Then $P \cap Q$ is a parabolic with Levi $L \cap M$ and for all $x \in P \cap Q$, one has $\theta_L^P(\theta_M^Q(x)) = \theta_M^Q(\theta_L^P(x)) = \theta_{L \cap M}^{P \cap Q}(x)$.*

Proof. Using the notation of Proposition 2.10 (3) we write $x = abcd \in P \cap Q$. Then $\theta_L^P(x) = ab$ and $\theta_M^Q(x) = ac$. This implies the first two assertions. The third assertion is obvious. The first part of the last assertion is Proposition 2.10 (2). Finally, write $x = \theta_L^P(x)u$ with $u \in R_u(P) \subseteq R_u(P \cap Q)$. Now $\theta_L^P(x) = \theta_M^Q(\theta_L^P(x))v$ with $v \in R_u(Q) \subseteq R_u(P \cap Q)$. The result follows. \square

We return to the situation of Subsection 2.3. Let

$$S = \{ (x, y) \in E ; x = y \} \subset E$$

be the scheme-theoretic stabilizer of $1 \in G$. Note that S is a finite group scheme that is usually not smooth. Let $S_0 := S_{\text{red}}$ be its underlying reduced subgroup scheme. Then S_0 is the finite constant group scheme over k such that $S_0(k) = S(k) \subset E(k)$. We will

often identify S_0 and $S_0(k)$. We view S as a subgroup of $P \cap Q$ via the first projection. In the sequel we will use $X^*(E)^S$ and $X^*(S)$. The next few results show that we can replace S by S_0 and give a description of S_0 . We define:

$$P_0 := \bigcap_{i \in \mathbb{Z}} P^{(q^i)} \quad Q_0 := \bigcap_{i \in \mathbb{Z}} Q^{(q^i)} \quad L_0 := \bigcap_{i \in \mathbb{Z}} L^{(q^i)}.$$

It is clear that P_0 and Q_0 are opposite parabolic subgroups of G , defined over \mathbb{F}_q , with common Levi subgroup L_0 . We have the following lemma:

Lemma 2.13. *One has $Q_0 \cap P \subset L$.*

Proof. This follows from Corollary 2.11, because $L_0 \cap R_u(P) = \{1\}$ and $R_u(Q_0) \cap R_u(P) = \{1\}$ (since $R_u(Q_0) \subset R_u(B)$ and $R_u(P) \subset R_u(B^-)$). \square

Lemma 2.14. *The group S is contained in $Q_0 \cap L$. The group S_0 is contained in L_0 . Further, one has $S_0 = L_0(\mathbb{F}_q)$ and $S \cap L_0 = S_0$.*

Proof. Let $x \in S$. By definition, one has $\varphi(\theta_L^P(x)) = \theta_M^Q(x)$. Since $x \in Q$, one has $\theta_L^P(x) \in Q$ by Corollary 2.12. It follows that $\theta_M^Q(x) \in \sigma(Q)$ and we deduce $x \in \sigma(Q)$ because $R_u(Q) \subset R_u(B) \subset \sigma(Q)$. Now we can apply the same argument to $\sigma(Q)$ to show $x \in \sigma^2(Q)$. Continuing this process, we get $x \in Q_0$. Since $S \subset P$, we also have $S \subset L$ by Lemma 2.13.

Now, to prove $S_0 \subset L_0$, we need only show that $S_0(k) \subset L_0(k)$, because S_0 is smooth. Let $x \in S_0(k)$. By definition, one has $\varphi(\theta_L^P(x)) = \theta_M^Q(x)$. Since $x \in P$, we have $\theta_M^Q(x) \in P$ by Corollary 2.12. We deduce that $\theta_L^P(x) \in \sigma^{-1}(P)$ and then $x \in \sigma^{-1}(P)$. Repeating the argument yields $x \in P_0$. We have showed that $S_0 \subset L_0$. Now for $x \in L_0$, x lies in S_0 if and only if $\varphi(x) = x$, so $S_0 = L_0(\mathbb{F}_q)$.

Since the Lang-Steinberg map $L_0 \rightarrow L_0$, $x \mapsto x^{-1}\varphi(x)$ is étale, it follows that the algebraic group $S \cap L_0 = \{x \in L_0, \varphi(x) = x\}$ is smooth, equal to the constant group $L_0(\mathbb{F}_q)$. So we deduce $S \cap L_0 = S_0$. \square

In the next lemma, we consider the map $\theta_{L_0}^{Q_0}: Q_0 \rightarrow L_0$.

Lemma 2.15. *If $x \in S$, then $\theta_{L_0}^{Q_0}(x) \in S_0$.*

Proof. Let $x \in S$, so by definition one has $\varphi(\theta_L^P(x)) = \theta_M^Q(x)$. Now $x \in L \cap Q_0$ by Lemma 2.14, so we have $\theta_L^P(x) = x$ and $\varphi(x) = \theta_M^Q(x)$. We deduce from this

$$\varphi(\theta_{L_0}^{Q_0}(x)) = \theta_{L_0}^{Q_0}(\varphi(x)) = \theta_{L_0}^{Q_0}(\theta_M^Q(x)) = \theta_M^Q(\theta_{L_0}^{Q_0}(x)) = \theta_{L_0}^{Q_0}(x)$$

since $L_0 \subset M$. We deduce $\theta_{L_0}^{Q_0}(x) \in L_0(\mathbb{F}_q) = S_0$. \square

Remark 2.16. The Levi subgroup L_0 is the group H_w of [PWZ1] 5.1 for $w = \eta \in {}^JW$ the longest element. Note that we chose the frame (T, B, g_0) in such a way that $g_0\eta = 1$.

Proposition 2.17. *We have $X^*(E)^S = X^*(E)^{S_0}$ and $X^*(S) = X^*(S_0)$.*

Proof. The first assertion follows from Lemma 2.15, because for all $x \in S$ and $\chi \in X^*(E)$, one has $\chi(x) = \chi(\theta_{L_0}^{Q_0}(x))$ so χ vanishes on S if and only if it does on S_0 .

For the second assertion, note that we already have proved $X^*(S) \subset X^*(S_0)$. We have an exact sequence

$$1 \rightarrow S^0 \rightarrow S \rightarrow \pi_0(S) \rightarrow 1$$

And since we are working over a perfect field, this exact sequence is split by the group $S_{red} = S_0$. So $S_0 \simeq \pi_0(S)$ and finally $X^*(S) = X^*(S_0)$. \square

Now, we can rewrite diagram (1.4) taking into account this new information :

$$(2.8) \quad \begin{array}{ccccccc} & & & & \frac{\mathcal{E}(U)}{X^*(G)} & & \\ & & & & \downarrow & & \\ & & & & Z^1(G)^E & \xrightarrow{=} & Z^1(G)^E \\ & & & & \downarrow \mathcal{L} & & \downarrow \\ 0 & \longrightarrow & X^*(G) & \longrightarrow & X^*(E) & \longrightarrow & \text{Pic}^E(G) \longrightarrow \text{Pic}_E(G) \longrightarrow 0 \\ & & \downarrow & & \downarrow = & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}(U) & \longrightarrow & X^*(E) & \longrightarrow & X^*(S_0) \longrightarrow \frac{X^*(S_0)}{\text{im}(X^*(E) \rightarrow X^*(S_0))} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Corollary 2.18. *The group $\text{Pic}^E(G)$ has no p -torsion.*

Proof. This follows immediately from the previous diagram, knowing that $X^*(S_0)$ has no p -torsion. \square

We will denote by ζ the group homomorphism

$$(2.9) \quad \zeta: X^*(L_0) \rightarrow X^*(L_0), \quad \chi \mapsto \chi - \chi \circ \varphi.$$

Remark 2.19. The map ζ is injective, because if $\chi = \chi \circ \varphi$, then there exists $d \geq 1$ such that $\chi = \chi \circ \varphi^d = \varphi^d \circ \chi$, and this implies $\chi = 1$.

Corollary 2.20. *Assume that $\text{Pic}(L_0) = 0$ (this is the case if $\text{Pic}(G) = 0$). Then we have an exact sequence*

$$0 \rightarrow X^*(L_0) \xrightarrow{\zeta} X^*(L_0) \rightarrow X^*(S_0) \rightarrow 0$$

In particular, the order of the group $X^(S_0)$ is equal to the absolute value of the determinant of ζ .*

Proof. This follows from Proposition 1.12, knowing that $S_0 = L_0(\mathbb{F}_q)$. \square

Corollary 2.21. *Assume G is split over \mathbb{F}_q and $\text{Pic}(L_0) = 0$. Then*

$$X^*(S_0) \simeq \left(\frac{\mathbb{Z}}{(q-1)\mathbb{Z}} \right)^d$$

with $d := \text{rk}(X^*(L_0))$.

Example 2.22. Let $G = GL_{n, \mathbb{F}_q}$, with the standard (upper) Borel pair (B, T) . Let $r \geq s \geq 0$ be two integers such that $r + s = n$. Let P be the parabolic of matrices of the form

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \text{ with } A \in GL_r(k), B \in GL_s(k), C \in M_{s,r}(k)$$

and define $Q := P^-$. Then $L_0 = L \simeq GL_r \times GL_s$ and $S_0 \simeq GL_r(\mathbb{F}_q) \times GL_s(\mathbb{F}_q)$. Since $\text{Pic}(GL_n) = 0$, we deduce that

$$X^*(S_0) = \left(\frac{\mathbb{Z}}{(q-1)\mathbb{Z}} \right)^2$$

In particular, $X^*(S_0)$ is killed by $q-1$.

Example 2.23. Let $G = U(n)$ be the unitary group associated to the $n \times n$ -matrix

$$J = \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & & & \\ 1 & & & \end{pmatrix}$$

In other words, for every \mathbb{F}_q -algebra, one has :

$$G(R) = \{ A \in GL_n(\mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} R), {}^t\sigma(A)JA = J \}$$

We make the identification $G_k = GL_{n,k}$. The Galois action is then given by $\sigma \cdot A = J^t\sigma(A)^{-1}J$ for all $A \in GL_n(k)$. Let (B, T) the standard (upper) Borel pair of $GL_{n,k}$. One sees immediately that it is defined over \mathbb{F}_q . Let $r \geq s \geq 0$ be two integers such that $r + s = n$ and let P and Q be the same parabolics as in the previous example. For an \mathbb{F}_q -algebra R , the group $L_0(R)$ is easily seen to be the set of elements in $G(R)$ of the form

$$\begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} \text{ with } A \in GL_s(R \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}), B \in GL_{r-s}(R \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}), C \in GL_s(R \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2})$$

with the conditions $\sigma^2(A) = A$, $C = {}^t\sigma(A)^{-1}$, and $B \in U(r-s)$. In other words, one has

$$L_0 \simeq U(r-s) \times \text{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(GL_s)$$

And it follows that

$$S_0 = L_0(\mathbb{F}_q) = U(r-s)(\mathbb{F}_q) \times GL_s(\mathbb{F}_{q^2})$$

For example if $n = 3$, it can be shown that one has $S \simeq S_0 \times \alpha_p$, so S is in general not reduced. Since $\text{Pic}(GL_n) = 0$, we can apply Corollary 2.20 so any group homomorphism $S_0 \rightarrow k^\times$ comes from a character of L_0 . We deduce that

$$X^*(S_0) = \frac{\mathbb{Z}}{(q+1)\mathbb{Z}} \times \frac{\mathbb{Z}}{(q^2-1)\mathbb{Z}}$$

In particular, $X^*(S_0)$ is killed by $q^2 - 1$.

2.6 Zip data attached to a cocharacter

In our applications to Shimura varieties, zip data will arise as follows. Let G be a reductive group defined over \mathbb{F}_q and let $\chi: \mathbb{G}_m \rightarrow G$ be a cocharacter of G defined over \mathbb{F}_{q^r} , $r \geq 1$. Then χ gives rise to an algebraic zip datum as follows. Let $P_\pm = P_\pm(\chi)$ be the pair of opposite parabolic subgroups of G defined by χ . The Lie algebra of P_+ (resp. P_-) is the sum of the weight spaces of weights ≤ 0 (resp. ≥ 0) in $\text{Lie}(G)$ under the action of χ . The parabolics P_\pm have a common Levi subgroup L which is the stabilizer of μ .

We obtain an algebraic zip datum $\mathcal{Z}_{G,\chi} := (G, P_+, \sigma(P_-), \varphi)$, where $\varphi: L \rightarrow L^{(q)}$ is the relative Frobenius. In particular we obtain an attached zip group $E := E_{G,\chi}$. This zip datum is of the form studied in Subsection 2.3.

This setting is particularly convenient with respect to functoriality. Indeed, if $\alpha: G \rightarrow G'$ is a morphism of reductive groups defined over \mathbb{F}_q , then a cocharacter $\mu: \mathbb{G}_m \rightarrow G$ (defined over a finite extension of \mathbb{F}_q) induces a cocharacter $\alpha \circ \mu$. Denote by E and E' the associated zip groups. Then $\alpha \times \alpha$ yields a homomorphism $E \rightarrow E'$ and we obtain an induced morphism of stacks

$$[E \backslash G] \rightarrow [E' \backslash G']$$

Zip data arising from small and minuscule cocharacters will be of great importance to us. Let us recall briefly the definitions. Let G be a reductive group with a Borel pair (T, B) and let $(X^*, \Phi, X_*, \Phi^\vee, \Delta)$ be the correspondig based root datum. Let G^{ad} be the adjoint group and let $G^{\text{ad}} = \prod_{i=1}^r G^{(i)}$ be its decomposition into simple adjoint groups. The image of (T, B) in $G^{(i)}$ is a Borel pair $(T^{(i)}, B^{(i)})$. We obtain a corresponding decomposition of the set of roots $\Phi = \coprod_{i=1}^r \Phi^{(i)}$, similarly for the set of coroots and the root basis Δ . Let us call a simple root $\alpha \in \Delta^{(i)}$ *special* if it lies in the orbit of the affine root under the action of the automorphism group of the extended Dynkin diagram of $(G^{(i)}, B^{(i)}, T^{(i)})$.

Definition 2.24. A cocharacter $\chi: \mathbb{G}_m \rightarrow G$ is called *minuscule* if it satisfies the following equivalent properties ([De] 1.2).

- (i) The representation $\text{ad} \circ \chi$ of \mathbb{G}_m on $\text{Lie}(G)$ has only weights $-1, 0$, and 1 .
- (ii) It is conjugate to a cocharacter χ' of T such that $\langle \chi', \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$.
- (iii) Let $\chi^{(i)}$ be the unique $B^{(i)}$ -dominant cocharacter of $T^{(i)}$ which is conjugate to the image of χ in $G^{(i)}$. Then there exists at most one simple root $\alpha \in \Delta^{(i)}$ such that $\langle \chi^{(i)}, \alpha \rangle > 0$ and in this case $\langle \chi^{(i)}, \alpha \rangle = 1$ and α is special.

Definition 2.25. A cocharacter $\chi: \mathbb{G}_m \rightarrow G$ is called *small* if it satisfies Condition (iii) of Definition 2.24 without the assumption that α is special.

Let $\chi \in X_*$ be a dominant small cocharacter and let $P = P_+(\chi)$ be the associated parabolic subgroup. Then the image $P^{(i)}$ of P in $G^{(i)}$ is either a maximal parabolic subgroup (if there exists α as in (iii)) or equal to $G^{(i)}$ (if $\chi^{(i)}$ is central in $G^{(i)}$).

3 Positivity

3.1 Ample characters and ample functions

Definition 3.1. Let X be a smooth variety and $U \subset X$ an open subset such that $X - U$ is purely of codimension 1. A function $f \in \mathcal{E}(U)$ will be called *U -ample* if it has an effective divisor, and if the support of the divisor of f is exactly the complement of U in G . If $f \in \mathcal{E}(U)_{\mathbb{Q}}$, we say that f is *U -ample* if some positive multiple of f in $\mathcal{E}(U)$ is *U -ample*.

When no confusion can occur about the open subset U , we will call a *U -ample* function simply an *ample* function.

Definition 3.2. Let G be a group and $H \subset G$ a closed subgroup. Let $\alpha \in X^*(H)$ be a character (see 1.1). We say that α is *G -ample* or simply *ample* if the line bundle $\xi(\alpha)$ on G/H defined by α is ample. Again we define $\alpha \in X^*(H)_{\mathbb{Q}}$ to be *G -ample* if some positive multiple in $X^*(H)$ is *G -ample*.

We define also an *antiample* character (resp. function) as the inverse of an ample character (resp. function).

Remark 3.3. Let G be a reductive group over an algebraically closed field k . Let (B, T) be a Borel pair and let $(X^*, \Phi, X_*, \Phi^\vee, \Delta)$ be the associated root datum and (W, I) be the Weyl group with its system of simple reflections. For $\alpha \in \Phi$ let $\alpha^\vee \in \Phi^\vee$ the dual root. For every $i \in I$ let $\alpha_i \in \Delta$ be the corresponding simple root and $\omega_i \in (X^*)_{\mathbb{Q}}$ the corresponding fundamental weight.

Let P_J be a parabolic subgroup containing B of type $J \subseteq I$ and let L be the unique Levi subgroup of P containing T . Then restriction yields an injective map $X^*(P_J) \cong X^*(L) \hookrightarrow X^* = X^*(T)$. Its image consists of those $\lambda \in X^*$ such that $\langle \alpha_j^\vee, \lambda \rangle = 0$ for all $j \in J$. It is well known (e.g. [Jan] II, 4.4) that for a character $\beta \in X^*(P_J)$ the following assertions are equivalent.

- (i) The character β is *G -ample*.
- (ii) One has $\langle \alpha_i^\vee, \beta \rangle > 0$ for all $i \in I \setminus J$.

In the sequel, P will always denote a parabolic subgroup of type J that contains the opposite Borel B_- of B with respect to T . Then $P_J = {}^{w_0}P$, where $w_0 \in W$ is the longest element. For $i \in I$, $-w_0(\alpha_i)$ is again in Δ . Hence the image of $X^*(P)$ in X^* consists of those λ such that $\langle \alpha_j^\vee, \lambda \rangle = 0$ for all $j \in J^{\text{opp}}$ and $\beta \in X^*(P)$ is *G -ample* if and only if $\langle \alpha_i^\vee, \beta \rangle < 0$ for all $i \in I \setminus J^{\text{opp}}$.

Remark 3.4. Assume G and P decompose as products $G = G_1 \times G_2$ and $P = P_1 \times P_2$. Let $\alpha \in X^*(P)$ be a character and let $\alpha = (\alpha_1, \alpha_2)$ be its decomposition. Then α is ample if and only if α_1 and α_2 are ample.

Remark 3.5. Assume $G \subset G_1$ and $P = P_1 \cap G$ for some parabolic P_1 in G_1 . Then we get a closed immersion $G/P \rightarrow G_1/P_1$. It follows that an ample character $\alpha \in X^*(P_1)$ restricts to an ample character of P .

3.2 The positivity conjecture

Now we study again a Frobenius zip datum as in Subsection 2.3. To formulate the next conjecture, recall that if $U \subset G$ denotes the open E -orbit of G , the injection $\iota: \mathcal{E}(U) \rightarrow X^*(E)$ has finite cokernel contained in $X^*(S_0)$, where S_0 is the underlying finite constant group scheme of the stabilizer of $1 \in G$ in the zip group E . In particular, it induces an isomorphism

$$\iota: \mathcal{E}(U)_{\mathbb{Q}} \xrightarrow{\sim} X^*(E)_{\mathbb{Q}}.$$

Furthermore, the first projection $p_1: E \rightarrow P$ induces an isomorphism $p_1^*: X^*(P) \simeq X^*(E)$.

Conjecture 3.6 (Positivity Conjecture). Let G be a reductive group defined over \mathbb{F}_q which has a Borel pair (B, T) defined over \mathbb{F}_q . Let P be a parabolic containing B^- and $Q := \sigma(P^-)$. Through the isomorphism

$$(3.1) \quad X^*(P)_{\mathbb{Q}} \xrightarrow{p_1^*} X^*(E)_{\mathbb{Q}} \xrightarrow{\iota^{-1}} \mathcal{E}(U)_{\mathbb{Q}}$$

an G -ample character $\alpha \in X^*(P)_{\mathbb{Q}}$ is mapped to an ample function in $\mathcal{E}(U)_{\mathbb{Q}}$.

Remark 3.7. Since the map $\mathcal{E}(U) \rightarrow Z^1(G)^E$ deduced from diagram (2.8) is $f \mapsto -\text{div}(f)$ (Remark 1.17), the conjecture asserts that an ample character in $X^*(P)_{\mathbb{Q}}$ is mapped to an element of $Z^1(G)_{\mathbb{Q}}^E$ with < 0 coefficients. Note also that this conjecture does not claim conversely that an ample function corresponds to an ample character via the isomorphism 3.1, which can be seen to be false in general, but true in the split case (see Remark 3.15).

We do not know whether the conjecture holds in general. For the applications to Shimura varieties we have in mind it suffices to prove the Positivity Conjecture for zip data defined by minuscule cocharacters (Subsection 2.6). In particular it will be sufficient to show the following theorem.

Theorem 3.8. *Suppose that the zip datum is defined by a small cocharacter (Definition 2.25) or that P and Q are defined over \mathbb{F}_q . Then Positivity Conjecture 3.6 holds.*

We will prove this theorem in the remaining subsections of Section 3. We first deal with the case that P and Q are defined over \mathbb{F}_q in Subsection 3.3. In the next two subsections we reduce to the case that G is of the scalar restriction of an adjoint simple group and make some remarks about embeddings of zip data. Finally, in Subsection 3.6 we prove the Positivity Conjecture in a setting that particularly holds if G is the scalar restriction of a simple adjoint group and if the zip datum is given by a small cocharacter.

3.3 The case where P and Q are defined over \mathbb{F}_q

The goal of this section is the proof of the Positivity Conjecture in full generality (i.e., without assumption that P is defined by a small cocharacter) but in the special case of a Frobenius zip datum as in Subsection 2.3 where the parabolics P and Q are both defined over \mathbb{F}_q . Hence we have $Q = P^-$ and $L = M$ is a common Levi subgroup of P and Q . As before we denote by $J \subseteq I$ the type of P and by $K = J^{\text{opp}}$ the type of K . We then have the following fact ([Wd2] Proposition 3.1):

Proposition 3.9. *The open $P \times Q$ -orbit in G coincides with the open E -orbit in G .*

In particular, one has $Z^1(G)^{P \times Q} = Z^1(G)^E$. Denoting by U the open E -orbit, one has a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(U)_{\mathbb{Q}} & \xrightarrow{j_1} & X^*(P \times Q)_{\mathbb{Q}} \\ & \searrow j_2 \simeq & \downarrow j \\ & & X^*(E)_{\mathbb{Q}} \end{array}$$

which induces a natural section $j = j_1 \circ j_2^{-1} : X^*(E)_{\mathbb{Q}} \rightarrow X^*(P \times Q)_{\mathbb{Q}}$ of the restriction map. To describe it concretely, consider the \mathbb{Q} -linear map

$$\zeta : X^*(L) \rightarrow X^*(L), \quad \chi \mapsto \chi - \chi \circ \varphi.$$

It induces an automorphism of $X^*(L)_{\mathbb{Q}}$ (see the argument in Remark 2.19). We will always make identifications $X^*(L)_{\mathbb{Q}} = X^*(P)_{\mathbb{Q}} = X^*(Q)_{\mathbb{Q}}$ via the inclusions $L \subset P$ and $L \subset Q$, and $X^*(E)_{\mathbb{Q}} = X^*(P)_{\mathbb{Q}}$ via the first projection $E \rightarrow P$.

Lemma 3.10. *There is a commutative diagram*

$$\begin{array}{ccc} X^*(E)_{\mathbb{Q}} & \xrightarrow{j} & X^*(P \times Q)_{\mathbb{Q}} \\ \downarrow \simeq & & \downarrow \simeq \\ X^*(L)_{\mathbb{Q}} & \longrightarrow & X^*(L)_{\mathbb{Q}} \times X^*(L)_{\mathbb{Q}} \end{array}$$

where the vertical maps are the above identifications and where the lower horizontal map is given by $\chi \mapsto (\zeta^{-1}(\chi), -\zeta^{-1}(\chi))$.

Proof. Let $\chi \in X^*(L)_{\mathbb{Q}}$. The corresponding function $f \in \mathcal{E}(U)_{\mathbb{Q}}$ satisfies $f(a^{-1}\varphi(a)) = \chi(a)f(1)$ for all $a \in L$. We want to determine the character $\chi' \in X^*(L \times L)_{\mathbb{Q}}$ such that $f(a^{-1}b) = \chi'(a, b)f(1)$ for all $(a, b) \in L \times L$. Writing $\chi' = (\chi'_1, \chi'_2) \in X^*(L)_{\mathbb{Q}} \times X^*(L)_{\mathbb{Q}}$, we get $f(a^{-1}b) = \chi'_1(a)\chi'_2(b)f(1)$. We see immediately that $\chi'_2(a) = \chi'_1(a)^{-1}$ and $\chi(a) = \chi'_1(a)\chi'_2(\varphi(a)) = \chi'_1(a)\chi'_1(\varphi(a))^{-1} = \zeta(\chi'_1(a))$. In other words, $(\chi'_1, \chi'_2) = (\zeta^{-1}(\chi), -\zeta^{-1}(\chi))$. \square

Remark 3.11. It follows that the image of the map $\mathcal{E}(U)_{\mathbb{Q}} \rightarrow X^*(P \times Q)_{\mathbb{Q}} = X^*(L)_{\mathbb{Q}} \times X^*(L)_{\mathbb{Q}}$ consists of elements of the form $(\alpha, -\alpha)$, $\alpha \in X^*(L)_{\mathbb{Q}}$.

Remark 3.12. If G is split, every character of P is defined over \mathbb{F}_q , so we deduce that $\zeta(\chi) = -(q-1)\chi$.

Lemma 3.13. *Let $\chi \in X^*(P)_{\mathbb{Q}}$ be a G -ample character. Then $\chi \circ \varphi$ is G -ample.*

Proof. As (B, T) are defined over \mathbb{F}_q , the Frobenius induces an automorphism on the Coxeter system (W, I) again denoted by φ . As Q is defined over \mathbb{F}_q , one has $\varphi(J^{\text{opp}}) = J^{\text{opp}}$. Hence the lemma follows from the characterisation of G -ample characters in Remark 3.3. \square

Lemma 3.14. *Let $\chi \in X^*(P)_{\mathbb{Q}}$ be an G -ample character. Then $\zeta^{-1}(\chi)$ is G -antiample.*

Proof. Write $\alpha = \zeta^{-1}(\chi)$, so $\chi = \alpha - \alpha \circ \varphi$. Choose $d \geq 1$ such that α is defined over \mathbb{F}_{q^d} . We deduce from Lemma 3.13 that

$$\chi + \chi \circ \varphi + \cdots + \chi \circ \varphi^d = \alpha - \alpha \circ \varphi^d = -(q^d - 1)\alpha$$

is G -ample, so $\alpha = \zeta^{-1}(\chi)$ is G -antiample. \square

Remark 3.15. However, if χ is a G -ample character, then it is not in general true that $\zeta(\chi)$ is G -antiample. This is the main reason why in Conjecture 3.6, we cannot expect to have an equivalence, but only one implication.

Denote by $p_2: X^*(P \times Q) \rightarrow X^*(Q)$ the projection onto the second factor. We have the following commutative diagram

$$(3.2) \quad \begin{array}{ccccc} & & X^*(P)_{\mathbb{Q}} & & \\ & & \downarrow \simeq & & \\ \mathcal{E}(U)_{\mathbb{Q}} & \xrightarrow{j_2} & X^*(E)_{\mathbb{Q}} & & \\ & \searrow j_1 & \downarrow j & & \\ & & X^*(P \times Q)_{\mathbb{Q}} & \xrightarrow{p_2} & X^*(Q)_{\mathbb{Q}} \\ \downarrow -\text{div} & & \downarrow & & \downarrow \\ Z_{\mathbb{Q}}^1(G)^{P \times Q} & \longrightarrow & \text{Pic}^{P \times Q}(G)_{\mathbb{Q}} & \longrightarrow & \text{Pic}(G/Q)_{\mathbb{Q}} \\ & & \searrow \delta & & \end{array}$$

In this diagram, the map $\text{Pic}^{P \times Q}(G)_{\mathbb{Q}} \rightarrow \text{Pic}(G/Q)_{\mathbb{Q}}$ is simply defined by identifying $\text{Pic}(G/Q)$ with $\text{Pic}^Q(G)$, and forgetting the P -action. By composition this defines a map $\delta: Z_{\mathbb{Q}}^1(G)^{P \times Q} \rightarrow \text{Pic}(G/Q)_{\mathbb{Q}}$.

Lemma 3.16. *The map $\delta: Z_{\mathbb{Q}}^1(G)^{P \times Q} \rightarrow \text{Pic}(G/Q)_{\mathbb{Q}}$ is an isomorphism. Further, if $C \in Z^1(G)^{P \times Q}$, then $\delta(C)$ is the line bundle attached to the Weil divisor $C/Q \subset G/Q$.*

Proof. Source and target of δ are \mathbb{Q} -vector spaces of the same dimension (equal to $\text{rk}(X^*(Q)) - \text{rk}(X^*(G))$), so we need only show that the map is injective. Let $C \in Z^1_{\mathbb{Q}}(G)^{P \times Q}$ be mapped to zero in $\text{Pic}(G/Q)_{\mathbb{Q}}$. The image of C in $\text{Pic}^{P \times Q}(G)_{\mathbb{Q}}$ has a preimage in $X^*(P \times Q)_{\mathbb{Q}} = X^*(L)_{\mathbb{Q}} \times X^*(L)_{\mathbb{Q}}$ of the form $(\alpha, 0)$. But C comes from a function $f \in \mathcal{E}(U)_{\mathbb{Q}}$, whose character in $X^*(P \times Q)_{\mathbb{Q}} = X^*(L)_{\mathbb{Q}} \times X^*(L)_{\mathbb{Q}}$ has the form $(\beta, -\beta)$, by Remark 3.11. So $\alpha = \beta = 0$, and $C = 0$.

Now consider the following commutative diagram :

$$\begin{array}{ccc} Z^1(G)^{P \times Q} & \longrightarrow & \text{Pic}^{P \times Q}(G) \\ \downarrow & & \downarrow \\ Z^1(G)^Q & \longrightarrow & \text{Pic}^Q(G) \xrightarrow{\cong} \text{Pic}(G/Q) \end{array}$$

The map $Z^1(G)^{P \times Q} \rightarrow Z^1(G)^Q$ is simply the inclusion (notice that $Z^1(G)^Q$ has infinite rank). Now the map $Z^1(G)^Q \rightarrow \text{Pic}(G/Q)$ is given by sending C to the line bundle attached to the Weil divisor C/Q , so we are done. \square

The Borel case Assume for a moment that P and Q are Borel subgroups, so $P = B^-$ and $Q = B$. For a simple root $\alpha \in \Delta$, denote by Z_{α} the closure of the $B^- \times B$ -orbit of \dot{s}_{α} . The Z_{α} for $\alpha \in \Delta$ form a basis of $Z^1(G)^{B^- \times B}$. The right- B -invariant subvariety $Z_{\alpha} \subset G$ defines the Weil divisor $\delta(Z_{\alpha}) = \overline{Z}_{\alpha} := Z_{\alpha}/B$ (Lemma 3.16). Let $\mathcal{L}_{\alpha} \in \text{Pic}(G/B)$ be the corresponding line bundle. Then one has the following ([LMS] page 99):

Proposition 3.17. *The natural map $X^*(B) \rightarrow \text{Pic}(G/B)$ is given by*

$$\chi \mapsto \sum_{\alpha \in \Delta} \langle \alpha^{\vee}, \chi \rangle \mathcal{L}_{\alpha}.$$

We claim that this proves Conjecture 3.6 in the Borel case. Indeed, take a character $\chi \in X^*(P) = X^*(B^-) = X^*(T)$ and suppose that χ is G -ample or, equivalently, that $\langle \alpha^{\vee}, \chi \rangle < 0$ for all $\alpha \in \Delta$ (Remark 3.3). Hence as an element of $X^*(Q) = X^*(B)$ the character χ is G -antiample. Consider the Diagram (3.2). By Lemma 3.10 the image of χ in $X^*(Q)_{\mathbb{Q}} = X^*(B)_{\mathbb{Q}}$ under $p_2 \circ j$ is given by $\gamma := -\zeta^{-1}(\chi)$. Lemma 3.14 applied to B shows that γ is G -antiample and hence that χ is mapped to an antiample line bundle $\mathcal{L}(\gamma) \in \text{Pic}(G/B)_{\mathbb{Q}}$. By Proposition 3.17 we have $\mathcal{L}(\gamma) = \sum_{\alpha \in \Delta} \langle \alpha^{\vee}, \gamma \rangle \mathcal{L}_{\alpha}$. But then $\delta^{-1}(L) = \sum_{\alpha} \langle \alpha^{\vee}, \gamma \rangle Z_{\alpha}$ by Lemma 3.16 and since γ is antiample, all the coefficients are < 0 .

The general case with P and Q defined over \mathbb{F}_q

Proposition 3.18. *In Conjecture 3.6, assume further that P is defined over \mathbb{F}_q . Then the conjecture holds.*

Proof. Again, we simply need to show that δ^{-1} maps an ample line bundle in $\text{Pic}(G/Q)$ to a linear combination with < 0 coefficients in $Z^1_{\mathbb{Q}}(G)^{P \times Q}$. We will deduce the result from the Borel case, using an elementary argument. Consider the following commutative diagram :

$$\begin{array}{ccccc}
Z^1(G)_{\mathbb{Q}}^{P \times Q} & \xrightarrow{\simeq} & \text{Pic}(G/Q)_{\mathbb{Q}} & \longleftarrow & X^*(Q)_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
Z^1(G)_{\mathbb{Q}}^{B^- \times B} & \xrightarrow{\simeq} & \text{Pic}(G/B)_{\mathbb{Q}} & \longleftarrow & X^*(B)_{\mathbb{Q}}
\end{array}$$

All the vertical maps are injective. The leftmost one is defined in the following way. If $C \subset G$ is a $P \times Q$ -orbit of codimension one, it is mapped to the only $B^- \times B$ -orbit of codimension one contained in it. Now, let $\chi \in X^*(Q)$ be an ample character. The restriction to B of χ is definitely not ample (unless $Q = B$), but it is a linear combination with ≤ 0 coefficients of the fundamental weights of the maximal parabolics containing B . Using the explicit formula available in the Borel case, we deduce that χ decomposes with ≤ 0 coefficients in $Z^1(G)^{B^- \times B}$ and thus also in $Z^1(G)^{P \times Q}$. Now, the number of simple roots β such that $\langle \chi, \beta^\vee \rangle < 0$ is equal to the cardinality of $I \setminus J$. This proves that all the $P \times Q$ -orbits of codimension one must appear with < 0 coefficients. \square

Example 3.19. Let $G = \text{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(GL_2, \mathbb{F}_{q^d})$. Let (B_0, T_0) be the usual upper Borel pair in GL_2, \mathbb{F}_{q^d} , and let (B, T) be its Weyl restriction to \mathbb{F}_q . In this example, we take $P = B_-$ and $Q = B$. The group G_k is isomorphic to a product of d copies of $GL_{2,k}$. Let $\alpha : T_0 \rightarrow k^\times$ be the character defined by

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mapsto a.$$

One checks easily that α is a GL_2 -antiample character of $B_{0,-}$. Let $\alpha_i = \alpha \circ \text{pr}_i$ where $\text{pr}_i : G_k \rightarrow GL_{2,k}$ is the i -th projection. The $(\alpha_i)_{1 \leq i \leq d}$ form a basis of $X^*(B)/X^*(G)$ and are the fundamental weights of the maximal parabolics containing B . The map ζ is given in this basis by

$$A = \begin{pmatrix} 1 & -q & & & \\ & 1 & \ddots & & \\ & & \ddots & -q & \\ -q & & & & 1 \end{pmatrix}$$

The inverse of A is the matrix

$$A^{-1} = -\frac{1}{q^d - 1} \begin{pmatrix} 1 & q & q^2 & \dots & q^{d-1} \\ q^{d-1} & 1 & q & \dots & q^{d-2} \\ \vdots & q^{d-1} & \ddots & \ddots & \vdots \\ q^2 & \vdots & \ddots & 1 & q \\ q & q^2 & \dots & q^{d-1} & 1 \end{pmatrix}$$

which has negative coefficients, as predicted by Lemma 3.14. The absolute value of the determinant of A is $q^d - 1$, which is the cardinality of $S_0 = \mathbb{F}_{q^d}^\times$, and this illustrates

Proposition 2.20. Let $[C_i] \in Z_{\mathbb{Q}}^1(G)^E$ be the closure of the E -orbit of $(1, \dots, J, \dots, 1)$ where $J = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Through the isomorphism

$$\frac{X^*(P)_{\mathbb{Q}}}{X^*(G)_{\mathbb{Q}}} \simeq \text{Pic}^E(G)_{\mathbb{Q}} \simeq Z_{\mathbb{Q}}^1(G)^E,$$

the cycle $-[C_i]$ corresponds to the character $\zeta(\alpha_i) = \alpha_i - p\alpha_{i-1}$ (where we define $\alpha_0 = \alpha_n$). This illustrates a result by Goren on Hasse invariants for Hilbert-Blumenthal Shimura varieties (see [Gor]).

3.4 Reduction to the case of a scalar restriction of an adjoint group

It is possible to reduce the proof of Theorem 3.8 to the case when the group G is semisimple adjoint. More precisely:

Proposition 3.20. *Keep the same notations as in Conjecture 3.6. Let $G' = G^{\text{ad}}$, and define T', B', P', Q' as the images of T, B, P, Q by the natural map $G \rightarrow G'$. Then (G', P', Q', φ) is a zip datum. Further, if Conjecture 3.6 holds for G' , then it does for G as well.*

Proof. The first assertion is obvious. Denote by E' the zip group of (G', P', Q', φ) . We get a map of stacks $[E \backslash G] \rightarrow [E' \backslash G']$. The E -orbits in G are in one-to-one correspondence with the E' -orbits in G' (2.4), so in particular the natural map $Z_{\mathbb{Q}}^1(G')^{E'} \rightarrow Z_{\mathbb{Q}}^1(G)^E$ is an isomorphism, and maps the canonical basis of $Z_{\mathbb{Q}}^1(G')^{E'}$ to the canonical basis of $Z_{\mathbb{Q}}^1(G)^E$. We have a commutative diagram :

$$\begin{array}{ccccccc} Z_{\mathbb{Q}}^1(G')^{E'} & \longrightarrow & \text{Pic}_{\mathbb{Q}}^{E'}(G') & \longleftarrow & X^*(E')_{\mathbb{Q}} & \longleftarrow & X^*(P')_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z_{\mathbb{Q}}^1(G)^E & \longrightarrow & \text{Pic}_{\mathbb{Q}}^E(G) & \longleftarrow & \frac{X^*(E)_{\mathbb{Q}}}{X^*(G)_{\mathbb{Q}}} & \longleftarrow & \frac{X^*(P)_{\mathbb{Q}}}{X^*(G)_{\mathbb{Q}}} \end{array}$$

where all maps are isomorphisms. A G -ample character of P is the image of a G' -ample character of P' (note that $G/P \simeq G'/P'$), and the result follows. \square

Now, assume that we have a direct product of two zip data. More precisely, this means that $G = G_1 \times G_2$ where G_1, G_2 are reductive groups over \mathbb{F}_q , and the groups T, B, P, Q decompose as $T = T_1 \times T_2$, $B = B_1 \times B_2$, $P = P_1 \times P_2$, $Q = Q_1 \times Q_2$, such that (G_1, P_1, Q_1, φ) and (G_2, P_2, Q_2, φ) are zip data.

Proposition 3.21. *Assume that Conjecture 3.6 holds for G_1 and G_2 . Then it holds for $G = G_1 \times G_2$.*

Proof. This follows immediately from Remark 3.4. \square

As every adjoint group over \mathbb{F}_q is isomorphic to a product of algebraic groups of the form $\text{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(G_0)$, where G_0 is a simple adjoint group over some finite extension \mathbb{F}_{q^r} of \mathbb{F}_q ([SGA3] Exp. XXIV, Prop. 5.5 and Prop. 5.9), this reduces the proof of Theorem 3.8 to the case that G is of this form.

3.5 Perfect embeddings of zip data

Let $G \subset G_1$ be two reductive groups defined over \mathbb{F}_q . Let (B, T) (resp. (B_1, T_1)) be a Borel pair defined over \mathbb{F}_q in G (resp. G_1). Let P (resp. P_1) be a parabolic in G (resp. G_1) containing B^- (resp. B_1^-). Define $Q := \sigma(P^-)$ and $Q_1 := \sigma(P_1^-)$. Let L, M, L_1, M_1 the associated Levi subgroups of P, Q, P_1, Q_1 respectively. Finally, let U (resp. U_1) denote the open E -orbit (resp. E_1 -orbit) in G (resp. G_1). We make the following assumptions :

1. $B = B_1 \cap G, T = T_1 \cap G, P = P_1 \cap G, Q = Q_1 \cap G, L = L_1 \cap G, M = M_1 \cap G, R_u(P) = R_u(P_1) \cap G, R_u(Q) = R_u(Q_1) \cap G.$
2. $U = U_1 \cap G.$

We call this situation a perfect embedding of zip data. In this case, the inclusion $P \times Q \rightarrow P_1 \times Q_1$ induces an inclusion $E \rightarrow E_1$, where E and E_1 are the corresponding zip groups. Furthermore, the inclusion $G \rightarrow G_1$ and the group actions of E and E_1 are compatible through this map. We get a map of quotient stacks $[E \backslash G] \rightarrow [E_1 \backslash G_1]$.

Lemma 3.22. *Assume that Conjecture 3.6 holds for G_1 . Assume further that the map $X^*(P_1) \rightarrow X^*(P)$ induces a surjection from the set of G -ample characters of P to the set of G_1 -ample characters of P_1 . Then Conjecture 3.6 holds for G .*

Proof. We have a commutative diagram :

$$\begin{array}{ccccc} X^*(P_1)_{\mathbb{Q}} & \xrightarrow{\cong} & X^*(E_1)_{\mathbb{Q}} & \xrightarrow{\cong} & \mathcal{E}(U_1)_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow \\ X^*(P)_{\mathbb{Q}} & \xrightarrow{\cong} & X^*(E)_{\mathbb{Q}} & \xrightarrow{\cong} & \mathcal{E}(U)_{\mathbb{Q}} \end{array}$$

It suffices to prove that an ample function $f_1 \in \mathcal{E}(U_1)$ is mapped to an ample function $f \in \mathcal{E}(U)$. It can be seen readily that f is the restriction of f_1 to U . Since $\text{div}(f_1)$ is an effective divisor, f_1 extends to a regular function on G_1 , so f extends to G . Thus $\text{div}(f)$ is effective. Now, f vanishes exactly on $G - (U_1 \cap G) = G - U$, so f is ample. \square

3.6 The case of a Weil restriction

We now prove the Positivity Conjecture in the following setting. Let $r \geq 1$ be an integer. Let G_1 be a connected reductive group over \mathbb{F}_{q^r} and $G = \text{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(G_1)$. Let k be an algebraic closure of \mathbb{F}_{q^r} . We will also write G_1 and G instead of $(G_1)_k$ and G_k . We denote by $\sigma \in \text{Gal}(k/\mathbb{F}_q)$ the q -th power arithmetic Frobenius. Fix a Borel pair (B_1, T_1) defined over \mathbb{F}_{q^r} in G_1 , and define $B = \text{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(B_1)$ and $T = \text{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(T_1)$. Then (B, T) is a Borel pair of G defined over \mathbb{F}_q . Over k , the group G decomposes as a product

$$(3.3) \quad G = G_1 \times \cdots \times G_r$$

where $G_i = \sigma^{i-1}(G_1)$. The Frobenius sends G_i onto G_{i+1} (indices taken modulo r). We have similar decompositions $T = T_1 \times \cdots \times T_r$ and $B = B_1 \times \cdots \times B_r$ over k , where (B_i, T_i) is a Borel pair of G_i .

Let $P \subseteq G$ be a parabolic subgroup containing B^- (defined over some finite extension of \mathbb{F}_q) that decomposes as a product

$$(3.4) \quad P = P_1 \times \cdots \times P_r$$

such that for each $i = 1, \dots, r$, the parabolic $P_i \subset G_i$ (defined over some finite extension of \mathbb{F}_{q^r}) is either maximal or equal to G_i . We have $B_i^- \subseteq P_i$. Let $\Delta \subseteq \{1, \dots, r\}$ be the subset where P_i is maximal in G_i .

The Weyl group (W, I) of G with respect to (T, B) decomposes naturally into a product (as a coxeter group)

$$(W, I) = (W_1, I_1) \times \cdots \times (W_r, I_r),$$

where (W_i, I_i) is the Weyl group of G_i with respect to (T_i, B_i) . The Frobenius induces an automorphism of Coxeter groups of (W, I) again denoted by σ and we have $\sigma(W_i, I_i) = (W_{i+1}, I_{i+1})$. In particular all factors are isomorphic. If J denotes the type of P , we have $W_J = W_{J_1} \times \cdots \times W_{J_r}$ where $J_i \subset I_i$ is the type of the parabolic P_i .

Let $T \subset L \subset P$ denote the Levi subgroup of P containing T and let P^- be the opposite parabolic subgroup with respect to L . Define the parabolic $Q \subset G$ by

$$Q := \sigma(P^-) = Q_1 \times \cdots \times Q_r, \quad Q_1 := \sigma(P_r^-), \quad Q_i := \sigma(P_{i-1}^-), 1 < i \leq r.$$

As $B = \sigma(B)$, we have $B \subseteq Q$. The Levi subgroup M of Q containing T is then $M = \sigma(L)$. We obtain a Frobenius zip datum (Subsection 2.3). Let E denote its zip group. Recall that the E -orbits are parametrized by ${}^J W$ through the bijection $w \mapsto O^w := E \cdot (g_0 w)$ where $g_0 = w_0 w_{0,J}$, and we have the relation $\dim O^w = \ell(w) + \dim(P)$. Thus the E -orbits of codimension 1 in G are :

$$(3.5) \quad C_j := E \cdot (1, \dots, (w_0 w_{0,J} s_j w_{0,J} w_0), \dots, 1), \quad j \in \Delta,$$

where $s_j \in I_j$ is the simple reflection such that $I_j \setminus J_j = \{s_j\}$.

Remark 3.23. An element of E can be written in the form

$$((x_1, \dots, x_r), (y_1, y_2, \dots, y_r))$$

with $x_i \in P_i$, $y_i \in Q_i$ and $\varphi(\bar{x}_i) = \bar{y}_{i+1}$ for all $1 \leq i \leq r$ (always taking modulo r).

3.6.1 First step

Define parabolic subgroups in G_1 by

$$\begin{aligned} P'_1 &:= P_1 \cap \sigma^{-1}(P_2) \cap \cdots \cap \sigma^{-(r-1)}(P_r), \\ Q'_1 &:= Q_1 \cap \sigma(Q_r) \cap \sigma^2(Q_{r-1}) \cap \cdots \cap \sigma^{r-1}(Q_2) \\ &= \sigma(P_r^-) \cap \sigma^2(P_{r-1}^-) \cap \cdots \cap \sigma^r(P_1^-). \end{aligned}$$

Since the Borel subgroup B is defined over \mathbb{F}_q , we see immediately that P'_1 and Q'_1 are parabolics in G_1 containing B_1^- and B_1 respectively. Their respective Levi subgroups with respect to the torus T_1 are

$$L'_1 := \bigcap_{i=1}^r \sigma^{-(i-1)}(L_i) \quad \text{and} \quad M'_1 := \bigcap_{i=1}^r \sigma^{r-i+1}(L_i) = \sigma^r(L'_1).$$

Since φ^r maps L'_1 to M'_1 , the quadruple $(G_1, P'_1, Q'_1, \varphi^r)$ is a Frobenius zip datum over \mathbb{F}_{q^r} . The associated zip group is given by

$$E'_1 := \{(x, y) \in P'_1 \times Q'_1, \varphi^r(\bar{x}) = \bar{y}\},$$

where $\bar{x} = \theta_{L'_1}^{P'_1}(x)$ and $\bar{y} = \theta_{L'_1}^{Q'_1}(y)$. It operates on G_1 in the usual way. For $(x, y) \in E'_1$ set

$$(3.6) \quad \begin{aligned} u_x &:= (x, \varphi(\bar{x}), \varphi^2(\bar{x}), \dots, \varphi^{r-1}(\bar{x})) \in P \\ v_y &:= (y, \varphi(\bar{x}), \varphi^2(\bar{x}), \dots, \varphi^{r-1}(\bar{x})) \in Q. \end{aligned}$$

It is clear that $(u_x, v_y) \in E$. The map $E'_1 \rightarrow E$, $(x, y) \mapsto (u_x, v_y)$ is an injective algebraic group homomorphism. The image of E'_1 is contained (in general strictly) in the stabilizer of G_1 in E which has the following description (considering G_1 as the subvariety $G_1 \times \{1\} \dots \{1\}$ of G).

Remark 3.24. Let $a = ((x_1, \dots, x_r), (y_1, y_2, \dots, y_r))$ be an element of E . Then the following assertions are equivalent.

- (i) $a(G_1) = G_1$.
- (ii) There exists $g_1 \in G_1$ such that $a \cdot g_1 \in G_1$.
- (iii) $x_i = y_i$ for all $1 < i \leq r$.

One has the following preliminary lemma.

Lemma 3.25. *Let $a = ((x_1, \dots, x_r), (y_1, y_2, \dots, y_r))$ be an element of $E(k)$ stabilizing G_1 . Then for all $1 \leq j \leq r$, one has*

$$(3.7) \quad x_j \in \tilde{P}_j := \bigcap_{i=j, j+1, \dots, r} \sigma^{j-i}(P_i)$$

Further, for all $j = 2, 3, \dots, r, r+1$, one has (taking indices modulo r)

$$(3.8) \quad y_j \in \tilde{Q}_j := \bigcap_{i=0, \dots, j-2} \sigma^i(Q_{j+2-i})$$

In particular, one has $x_1 \in \tilde{P}_1 = P'_1$ and $y_1 \in \tilde{Q}_1 = Q'_1$. Further, one has $(x_1, y_1) \in E'_1(k)$.

Proof. By Remark 3.24 we have $x_i = y_i$ for all $1 < i \leq r$. By definition, one has

$$(*) \quad \varphi\left(\theta_{L'_j}^{P'_j}(x_j)\right) = \theta_{M'_{j+1}}^{Q'_{j+1}}(y_{j+1}).$$

Now, let us fix an integer $1 \leq j < r$ and let us prove the first asertion using decreasing induction on the integer j . For $j = r$ the assertion is $x_r \in P_r$ and there is nothing to prove. By induction we have

$$y_{j+1} = x_{j+1} \in \tilde{P}_{j+1} = \bigcap_{i=j+1}^r \sigma^{-(i-j-1)}(P_i)$$

Now we apply Corollary 2.12 (ii) to the parabolics \tilde{P}_{j+1} and Q_{j+1} in G_{j+1} (both containing the torus T_{j+1}). We get

$$\theta_{M_{j+1}}^{Q_{j+1}}(y_{j+1}) \in \tilde{P}_{j+1}.$$

From (*) we deduce $\theta_{L_j}^{P_j}(x_j) \in \bigcap_{i=j}^r \sigma^{-(i-j)}(P_i) = \tilde{P}_j$ (because we consider only k -valued points). Since $\tilde{P}_j \subseteq P_j$ both contain the same Borel B_j^- , the unipotent radical of P_j is contained in the unipotent radical of \tilde{P}_j . Hence we find $x_j \in \tilde{P}_j$ which proves (3.7).

A similar argument shows (3.8): The proof is by increasing induction on j on the total order $2 < 3 < \dots < r < 1$. For $j = 2$ the assertion is $y_2 \in Q_2$. Now let $j > 2$. We have $x_{j-1} = y_{j-1} \in \tilde{Q}_{j-1} \cap P_{j-1}$ by induction hypothesis. Again we apply Corollary 2.12 (ii) to deduce $\theta_{L_{j-1}}^{P_{j-1}} \in \tilde{Q}_{j-1}$ and hence $\theta_{M_j}^{Q_j}(y_j) \in Q_j \cap \sigma(\tilde{Q}_{j-1}) = \tilde{Q}_j$ by (*). As the unipotent radical of Q_j is contained in the unipotent radical of Q_j , we find $y_j \in \tilde{Q}_j$.

It remains to prove that $\varphi^r(\theta_{L'_1}^{P'_1}(x_1)) = \theta_{M'_1}^{Q'_1}(y_1)$. For this, we use induction to prove:

$$(**) \quad \varphi^j \left(\theta_{\sigma^{-(j-1)}L_j}^{\sigma^{-(j-1)}P_j} \theta_{\sigma^{-(j-2)}L_{j-1}}^{\sigma^{-(j-2)}P_{j-1}} \dots \theta_{L_1}^{P_1}(x_1) \right) = \theta_{\sigma^{j-1}M_2}^{\sigma^{j-1}Q_2} \dots \theta_{\sigma M_j}^{\sigma Q_j} \theta_{M_{j+1}}^{Q_{j+1}}(y_{j+1})$$

For $j = 1$ this is clear. Assume that (**) holds and apply the operator $\varphi \circ \theta_{L_{j+1}}^{P_{j+1}}$ to (**). We get:

$$\begin{aligned} \varphi^{j+1} \left(\theta_{\sigma^{-j}L_{j+1}}^{\sigma^{-j}P_{j+1}} \theta_{\sigma^{-(j-1)}L_j}^{\sigma^{-(j-1)}P_j} \dots \theta_{L_1}^{P_1}(x_1) \right) &= \varphi \left(\theta_{L_{j+1}}^{P_{j+1}} \theta_{\sigma^{j-1}M_2}^{\sigma^{j-1}Q_2} \dots \theta_{\sigma M_j}^{\sigma Q_j} \theta_{M_{j+1}}^{Q_{j+1}}(y_{j+1}) \right) \\ &= \varphi \left(\theta_{\sigma^{j-1}M_2}^{\sigma^{j-1}Q_2} \dots \theta_{\sigma M_j}^{\sigma Q_j} \theta_{M_{j+1}}^{Q_{j+1}} \theta_{L_{j+1}}^{P_{j+1}}(y_{j+1}) \right) \\ &= \theta_{\sigma^j M_2}^{\sigma^j Q_2} \dots \theta_{\sigma^2 M_j}^{\sigma^2 Q_j} \theta_{\sigma M_{j+1}}^{\sigma Q_{j+1}} \varphi \left(\theta_{L_{j+1}}^{P_{j+1}}(x_{j+1}) \right) \\ &= \theta_{\sigma^j M_2}^{\sigma^j Q_2} \dots \theta_{\sigma^2 M_j}^{\sigma^2 Q_j} \theta_{\sigma M_{j+1}}^{\sigma Q_{j+1}} \theta_{M_{j+2}}^{Q_{j+2}}(y_{j+2}) \end{aligned}$$

which is (**) for $j + 1$. The above holds also for $j = r - 1$ if we take indices modulo r . Now (**) for $j = r$ together with Corollary 2.12 (iv) gives exactly $\varphi^r(\theta_{L'_1}^{P'_1}(x_1)) = \theta_{M'_1}^{Q'_1}(y_1)$ as claimed. \square

Lemma 3.26. *Let X_1 be the set of E -orbits in G intersecting G_1 . Then the map $\sigma \mapsto \sigma \cap G_1$ defines a bijection between X_1 and the set of E'_1 -orbits in G_1 .*

Proof. First we need to prove that $\sigma \cap G_1$ is an E'_1 -orbit. Let $u, v \in \sigma \cap G_1$. We can find an element

$$a = ((x_1, \dots, x_r), (y_1, \dots, y_r)) \in E$$

such that $a \cdot u = v$. But then a is in the stabilizer of G_1 (Remark 3.24), so $(x_1, y_1) \in E'_1$ by Lemma 3.25. Thus $\sigma \cap G_1$ is contained in a E'_1 -orbit.

Now, let $g \in \sigma \cap G_1$ and $(x, y) \in E'_1$. Define $\bar{x} := \theta_{L'_1}^{P'_1}(x)$ and $\bar{y} := \theta_{L'_1}^{Q'_1}(y)$. Consider the pair $(u_x, v_y) \in E$ defined as in (3.6). Then $xgy^{-1} = u_xgv_y^{-1}$, so $\sigma \cap G_1$ is exactly an E'_1 -orbit. This also shows the injectivity of the map $\sigma \mapsto \sigma \cap G_1$. The surjectivity is clear. \square

Lemma 3.27. *Let X be a regular integral noetherian scheme, let $Y \subset X$ be an integral subscheme of codimension 1, and let $Z \subseteq G$ be an irreducible subscheme of X not contained in Y . Then every irreducible component of $Z \cap Y$ has codimension 1 in Z .*

Proof. Since X is regular, Y is a Cartier divisor. Hence the open immersion $X \setminus Y \rightarrow X$ is an affine morphism. Therefore the open immersion $j: (X \setminus Y) \cap Z \rightarrow Z$ is affine. As Z is not contained in Y and Z is irreducible, j is dominant. Hence every irreducible component of $Z \setminus ((X \setminus Y) \cap Z) = Z \cap Y$ has codimension 1 in Z by [WdYa] Proposition 1.6. \square

We may define also the groups P'_j, Q'_j, E'_j for any $1 \leq j \leq r$. Note that we can permute the factors to form a new product

$$G_j \times G_{j+1} \times \cdots \times G_r \times G_1 \times \cdots \times G_{j-1}$$

and define P'_j, Q'_j, E'_j as before with respect to this new numbering. In other words, we define :

$$P'_j = P_j \cap \sigma^{-1}(P_{j+1}) \cap \cdots \cap \sigma^{-(r-j)}(P_r) \cap \sigma^{-(r-j+1)}(P_1) \cap \cdots \cap \sigma^{-(r-1)}(P_{j-1})$$

$$Q'_j = \sigma(P_j^-) \cap \sigma^{r-1}(P_{j+1}^-) \cap \cdots \cap \sigma^j(P_r^-) \cap \sigma^{j-1}(P_1^-) \cap \cdots \cap \sigma(P_{j-1}^-)$$

The tuple $(G_j, P'_j, Q'_j, \varphi^r)$ defines a group E'_j as usual. Then there is a one-to-one correspondance between the E -orbits in G intersecting G_j with the set of E'_j -orbits in G_j . This bijection is defined by intersecting an E -orbit with G_j . Let $(x, y) \in E'_j$ and write $\bar{x} := \theta_{L'_j}^{P'_j}(x)$ and $\bar{y} := \theta_{L'_j}^{Q'_j}(y)$. For all $1 \leq j \leq r$, let U_j denote the open E'_j -orbit in G_j . Then it is clear that $U_j = U \cap G_j$ (it is the E'_j -orbit of $1 \in G_j$). Define :

$$u := (\varphi^{r-j+1}(\bar{x}), \dots, \varphi^{r-1}(\bar{x}), x, \varphi(\bar{x}), \dots, \varphi^{r-j}(\bar{x})) \in P$$

$$v := (\varphi^{r-j+1}(\bar{x}), \dots, \varphi^{r-1}(\bar{x}), y, \varphi(\bar{x}), \dots, \varphi^{r-j}(\bar{x})) \in Q$$

It is clear that $(u, v) \in E$. This defines an embedding $\gamma_j: E'_j \rightarrow E$.

Remark 3.28. If P_1, \dots, P_r are all defined over \mathbb{F}_{q^r} , then the groups E'_j are simply Galois translates of each other. In particular, the number of E -orbits intersecting G_j is in this case independant of j .

3.6.2 Second step

We may define a map $\gamma_j : P'_j \rightarrow P$ compatible with the map $\gamma_j : E'_j \rightarrow E$ defined above. For $x \in P'_j$, define:

$$\gamma_j(x) = (\varphi^{r-j+1}(\bar{x}), \dots, \varphi^{r-1}(\bar{x}), x, \varphi(\bar{x}), \dots, \varphi^{r-j}(\bar{x}))$$

where $\bar{x} := \theta_{L'_j}^{P'_j}(x)$. We will now construct a commutative diagram:

$$(3.9) \quad \begin{array}{ccccccc} X^*(P)_{\mathbb{Q}} & \xrightarrow{\simeq} & X^*(E)_{\mathbb{Q}} & \xrightarrow{\simeq} & \mathcal{E}(U)_{\mathbb{Q}} & \xrightarrow{-\text{div}} & Z^1(G)_{\mathbb{Q}}^E \\ \downarrow \gamma_j^* & & \downarrow \gamma_j^* & & \downarrow \iota_j^* & & \downarrow u_j \\ X^*(P'_j)_{\mathbb{Q}} & \xrightarrow{\simeq} & X^*(E'_j)_{\mathbb{Q}} & \xrightarrow{\simeq} & \mathcal{E}(U_j)_{\mathbb{Q}} & \xrightarrow{-\text{div}} & Z^1(G_j)_{\mathbb{Q}}^{E'_j} \end{array}$$

The maps γ_j^* is the composition with the embedding γ_j . If $f \in \mathcal{E}(U)$, then $\iota_j^*(f)$ is the restriction of f to U_j via the natural embedding $\iota_j : G_j \rightarrow G$, $x \mapsto (1, \dots, x, \dots, 1)$.

Lemma 3.29. *The map ι_j^* extends uniquely to a map $u_j : Z^1(G)_{\mathbb{Q}}^E \rightarrow Z^1(G_j)_{\mathbb{Q}}^{E'_j}$.*

Proof. The map u_j is clearly unique. Now if $f \in \mathcal{E}(U)$ is the restriction of a character $\chi \in X^*(G)$, then $f \circ \iota_j$ is the restriction of the character $\chi \circ \iota_j$ of G_j , so we may define $u_j(\text{div}(f)) = \text{div}(f \circ \iota_j)$ for all $f \in \mathcal{E}(U)_{\mathbb{Q}}$. \square

If C is an irreducible component of $G - U$, then we can write $[C] = \text{div}(f)$ for some $f \in \mathcal{E}(U)_{\mathbb{Q}}$. So f extends to a non-vanishing function on $G - C$. The intersection $C \cap G_j$ is empty or has codimension one in G_j (Lemma 3.27). We conclude that the divisor of $f \circ \iota_j$ has support $C \cap G_j$ if this intersection is nonempty. In any case we may write

$$(3.10) \quad u_j([C]) = a_j(C) \cdot [C \cap G_j], \quad a_j(C) \geq 0$$

and $a_j(C) > 0$ if and only if $C \cap G_j \neq \emptyset$.

Lemma 3.30. *The diagram 3.9 is commutative.*

Proof. For $f \in \mathcal{E}(U)$, the corresponding character $\chi \in X^*(E)$ is defined by the formula $f(e \cdot x) = \chi(e)^{-1} f(x)$ for all $e \in E$ and $x \in U$. Now take $x = \iota_j(y)$ and $e = \gamma_j(u)$ with $y \in U_j$ and $u \in E'_j$. Since $\gamma_j(u) \cdot \iota_j(y) = \iota_j(u \cdot y)$, we get $f(\iota_j(u \cdot y)) = \chi(\gamma_j(u))^{-1} f(\iota_j(y))$ and the result follows. \square

Lemma 3.31. *Assume $\chi \in X^*(P)_{\mathbb{Q}}$ is G -ample. Then $\gamma_j^*(\chi) \in X^*(P'_j)_{\mathbb{Q}}$ is G_j -ample.*

Proof. To simplify notations, we will assume $j = 1$. Recall that $P'_1 := \bigcap_{i=1}^r \sigma^{-(i-1)}(P_i)$, so the maximal parabolics of G_1 containing P'_1 are the $\sigma^{-(i-1)}(P_i)$ for $i \in \Delta$ (some of them may be equal). For all $i \in \Delta$, let $\chi_i \in X^*(P_i)_{\mathbb{Q}}$ be the fundamental weight of P_i . Then $\chi = \sum_{i \in \Delta} a_i \chi_i$ with $a_i < 0$ (see Remark 3.3). We deduce that for all $x \in P'_1$, one has:

$$\chi \circ \gamma_1 = \sum_{i \in \Delta} a_i (\chi_i \circ \varphi^{i-1}) = \sum_{i \in \Delta} a_i q^{i-1} \chi_i^{\sigma^{-(i-1)}}$$

where $\chi_i^{\sigma^{-(i-1)}}$ is the fundamental weight of $\sigma^{-(i-1)}(P_i)$. It follows that $\chi \circ \gamma_1$ is a linear combination with < 0 coefficients of the fundamental weights of the maximal parabolics containing P'_1 , so it is ample. \square

Lemma 3.32. *Let $D \in Z^1(G)_{\mathbb{Q}}^E$. Assume that $u_j(D) > 0$ for all $1 \leq j \leq r$. Then $D > 0$.*

Proof. Write $D = \sum_{i \in \Delta} n_i [\overline{C}_i]$ where C_i are the codimension one E -orbits (see equation 3.5) and \overline{C}_i is its closure. Fix an integer $j \in \Delta$. Define $a_{i,j} := a_j(\overline{C}_i)$ (see equation 3.10). Now we have

$$u_j(D) = \sum_{i \in \Delta} a_{i,j} n_i [\overline{C}_i \cap G_j]$$

Clearly $a_{j,j} > 0$ because C_j intersects G_j . To prove the claim, it suffices to show that in this sum, the only $i \in \Delta$ contributing to $[\overline{C}_j \cap G_j]$ is $i = j$. So assume that $\overline{C}_j \cap G_j = \overline{C}_i \cap G_j$, $i \in \Delta$. Since the intersection $C_j \cap G_j$ is nonempty, it follows that $C_j \cap \overline{C}_i \neq \emptyset$, so $C_j \subset \overline{C}_i$ because C_j is an E -orbit. Looking at dimensions, it follows that $C_j = C_i$, so $i = j$. From $u_j(D) > 0$ we deduce $n_j > 0$ and finally $D > 0$. \square

Proposition 3.33. *Assume Conjecture 3.6 holds for the zip datum $(G_j, P'_j, Q'_j, \varphi^r)$ for all $1 \leq j \leq r$. Then it holds for (G, P, Q, φ) .*

Proof. This is a simple application of the previous lemmas using the commutativity of diagram 3.9. \square

Corollary 3.34. *Assume P_1, \dots, P_r are defined over \mathbb{F}_{q^r} . Then Theorem 3.6 holds.*

Proof. Assume that P_1, \dots, P_r are defined over \mathbb{F}_{q^r} . Then P'_j and Q'_j are defined over \mathbb{F}_{q^r} , so the result comes from Proposition 3.18. \square

3.6.3 Third step

In this third step we assume that P_1, \dots, P_r are defined over some finite field $\mathbb{F}_{q^{rd}}$. We consider the group

$$\tilde{G} = \text{Res}_{\mathbb{F}_{q^{rd}}/\mathbb{F}_q}(G_1 \times_{\mathbb{F}_{q^r}} \mathbb{F}_{q^{rd}})$$

We embed $G = \text{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(G_1)$ into \tilde{G} , such that for any \mathbb{F}_q -algebra R , the map $G(R) \rightarrow \tilde{G}(R)$ is given by the natural map $R \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \rightarrow R \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{rd}}$ induced by the inclusion $\mathbb{F}_{q^r} \subset \mathbb{F}_{q^{rd}}$. Over the algebraic closure, it decomposes into a product

$$\tilde{G}_k = (G_1 \times \dots \times G_r) \times \dots \times (G_1 \times \dots \times G_r)$$

where the product $G = G_1 \times \dots \times G_r$ appears d times. Note that the groups \tilde{G} , $G \times \dots \times G$ and $\text{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(G)$ are all different, although they become isomorphic over k .

The Galois action on $\tilde{G}(k)$ is given by

$$\sigma \cdot (x_1, \dots, x_{rd}) = (\sigma \cdot x_{rd}, \sigma \cdot x_1, \dots, \sigma \cdot x_{rd-1}).$$

Define $\tilde{P} = P \times \cdots \times P$ and $\tilde{Q} = Q \times \cdots \times Q$ inside $\tilde{G}_k = G_k \times \cdots \times G_k$, and denote by \tilde{E} the associated zip group.

Lemma 3.35. *This defines a perfect embedding of zip data.*

Proof. The only non trivial part is to show that $U = \tilde{U} \cap G$ where \tilde{U} is the open orbit in \tilde{G} . For this, it suffices to prove that the embedding $G \rightarrow \tilde{G}$ induces an injection from the set of E -orbits in G to the set of \tilde{E} -orbits in \tilde{G} . But these sets are parametrized respectively by JW and $\tilde{{}^JW} := {}^JW \times \cdots \times {}^JW$, and the map ${}^JW \rightarrow \tilde{{}^JW}$ is the diagonal embedding, so we are done. \square

Conjecture 3.6 holds for \tilde{G} (see Corollary 3.34). It is clear that the map $X^*(\tilde{P}) \rightarrow X^*(P)$ defines a surjection on the ample characters of \tilde{P} and P . It follows from Lemma 3.22 that Conjecture 3.6 also holds for G . This terminates the proof.

4 Hasse invariants for Shimura varieties of Hodge type

4.1 Shimura varieties of Hodge type

Let (\mathbf{G}, \mathbf{X}) be a Shimura datum. Hence \mathbf{G} is a reductive group over \mathbb{Q} and \mathbf{X} is a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{R}}$ satisfying Deligne's conditions ([Del]). We denote by $[\mu]$ the $\mathbf{G}(\mathbb{C})$ -conjugacy class of the component of $h_{\mathbb{C}}: \prod_{\text{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}$ corresponding to $\text{id} \in \text{Gal}(\mathbb{C}/\mathbb{R})$. The elements of $[\mu]$ are minuscule by Deligne's axioms. The field of definition of $[\mu]$ is a finite extension E of \mathbb{Q} , the reflex field.

We assume that (\mathbf{G}, \mathbf{X}) is of Hodge type, i.e., it can be embedded into a Shimura datum of the form $(\text{GSp}(V), S^{\pm})$, where $V = (V, \psi)$ is a symplectic space over \mathbb{Q} and where S^{\pm} is the double Siegel half space. We choose such an embedding ι .

Let p be a prime number such that \mathbf{G} has a reductive model \mathcal{G} over $\mathbb{Z}_{(p)}$ (equivalent, $\mathbf{G}_{\mathbb{Q}_p}$ has a reductive model over \mathbb{Z}_p [Ki1] (2.3.2) which we also denote by \mathcal{G}). Hence $K_p := \mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$. We denote by G the special fiber of \mathcal{G} . Hence G is a reductive group over \mathbb{F}_p .

Choose a place v of the reflex field E of (\mathbf{G}, \mathbf{X}) over p . Let $K = K_p K^p \subseteq \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup. If K^p is sufficiently small (which we assume from now on), Kisin ([Ki1]) and Vasiu ([Va]) have shown the existence of integral canonical models $\mathcal{S}_K(\mathbf{G}, \mathbf{X})$ for the Shimura variety attached to (\mathbf{G}, \mathbf{X}) and K (with restrictions for $p = 2$) over $O_{E, v}$. Here we follow Kisin and hence assume that for $p = 2$ Condition (2.3.4) of [Ki1] is satisfied. In particular, \mathbf{G}^{ad} has no factor of Dynkin type B for $p = 2$. We denote by $S := \mathcal{S}_K(\mathbf{G}, \mathbf{X})$ the special fiber. It is a smooth quasi-projective scheme over $\kappa := \kappa(v)$ the residue field of the place v .

There exists a $\mathbb{Z}_{(p)}$ -lattice Λ of V such that the embedding ι in the Siegel Shimura datum is induced by an embedding $G_{\mathbb{Z}_{(p)}} \rightarrow \text{GL}(\Lambda)$ ([Ki1] Lemma (2.3.1); see also the remarks in the proof of Lemma 4.6.2 in [Per]). By Zarhin's trick we may assume after changing (V, ψ) and Λ that ψ induces a perfect $\mathbb{Z}_{(p)}$ -pairing on Λ ([Ki2] (1.3.3)). We obtain an embedding

$$(4.1) \quad \iota: \mathcal{G} \hookrightarrow \text{GSp}(\Lambda)$$

of reductive group schemes over $\mathbb{Z}_{(p)}$ whose generic fiber is an embedding of Shimura data. We call such an embedding a *p-integral Hodge embedding*.

By [Ki1] 1.3.2, ι identifies \mathcal{G} with the scheme theoretic stabilizer of a finite set s of tensors in Λ^\otimes . Here for a finite locally free module M over a ring we write M^\otimes for the direct sum of all R -modules that one obtains from M by applying the operations of taking duals, tensor products, symmetric powers and exterior powers finitely often. Then we can identify Λ^\otimes with $(\Lambda^*)^\otimes$. Moreover we identify $\mathrm{GL}(\Lambda)$ with $\mathrm{GL}(\Lambda^*)$ via $g \mapsto g^\vee := {}^t g^{-1}$ and hence

$$\mathcal{G} = \{ g \in \mathrm{GL}(\Lambda^*) ; g^\vee(s) = s \}.$$

We set $\tilde{K}_p := \mathrm{GSp}(\Lambda)(\mathbb{Z}_p)$. By [Ki1] (2.1.2) there exists for K^p sufficiently small an open compact subgroup $\tilde{K}^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ containing K^p such that ι yields an embedding

$$\varepsilon^0 : \mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \hookrightarrow \mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^\pm),$$

where $\tilde{K} := \tilde{K}_p \tilde{K}^p$. The left hand side can be identified with a moduli spaces of polarized abelian varieties. More precisely, for \tilde{K}^p sufficiently small, let

$$\tilde{\mathcal{S}} := \mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm)$$

be the smooth quasi-projective $\mathbb{Z}_{(p)}$ -scheme whose T -valued points (T a $\mathbb{Z}_{(p)}$ -scheme) the set of isomorphism classes $(\mathcal{A}, \lambda, \eta)$, where

- (a) \mathcal{A} is an abelian scheme over T up to prime to p isogeny,
- (b) λ is an equivalence class of a prime to p quasi-isogeny $\lambda : \mathcal{A} \rightarrow \mathcal{A}^\vee$ such that locally on T some multiple is a polarization, and where two such quasi-isogenies are equivalent if they differ by a global section of the constant sheaf with value $\mathbb{Z}_{(p)}^\times$ on T ,
- (c) η is a \tilde{K}^p -level structure, i.e. a section over T of $\underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \hat{V}^p(\mathcal{A}))/\tilde{K}^p$, where

$$\hat{V}^p(\mathcal{A}) := (\lim_{p \nmid n} \mathcal{A}[n])_{\mathbb{Q}}.$$

Then the generic fiber of $\mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm)$ is naturally identified with $\mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(V), S^\pm)$ and the integral canonical model $\mathcal{S}_K(\mathbf{G}, \mathbf{X})$ is defined as the normalization of the closure of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ in $\mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,v}$ ([Ki2] (1.3.4)). In particular, one obtains a finite morphism of $\mathcal{O}_{E,v}$ -schemes

$$(4.2) \quad \varepsilon : \mathcal{S} := \mathcal{S}_K(\mathbf{G}, \mathbf{X}) \rightarrow \mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,v}$$

extending ε^0 .

The conjugacy class of cocharacters $[\mu]$ defines a conjugacy class of cocharacters of \mathcal{G} (resp. of G) defined over \mathcal{O}_{E_v} (resp. over κ) which we again denote by $[\mu]$. As \mathcal{G} is quasi-split, there exists a representative in $[\mu]$ defined over \mathcal{O}_{E_v} (resp. over κ).

4.2 The De Rham cohomology and the Hodge line bundle

Let $\tilde{\mathcal{A}} \rightarrow \mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^{\pm})$ be the universal abelian scheme. We call its pullback \mathcal{A} to $\mathcal{S} := \mathcal{S}_K(\mathbf{G}, \mathbf{X})$ via ε (4.2) the universal abelian scheme over $\mathcal{S}_K(\mathbf{G}, \mathbf{X})$. We set

$$(4.3) \quad \mathcal{V}_{\mathcal{S}} := H_{\mathrm{DR}}^1(\mathcal{A}/\mathcal{S})$$

This is a locally free $\mathcal{O}_{\mathcal{S}}$ -module canonically endowed with a finite set s_{dR} of sections of $\mathcal{V}_{\mathcal{S}}^{\otimes}$ that are horizontal with respect to the Gauß-Manin connection.

Then $e^*\Omega_{\mathcal{A}/\mathcal{S}}^1$ (where e is the zero section of \mathcal{A}) is a locally direct summand of $\mathcal{V}_{\mathcal{S}}$, the *Hodge filtration*. We call the highest exterior power of the Hodge filtration

$$\omega_{\mathcal{S}} := \det(e^*\Omega_{\mathcal{A}/\mathcal{S}}^1)$$

the *Hodge line bundle on \mathcal{S}* . It depends on the chosen p -integral Hodge embedding ι . In case we want to stress this dependency, we write $\omega_{\mathcal{S}}(\iota)$.

Proposition 4.1. *The line bundle $\omega_{\mathcal{S}}$ is ample.*

Proof. By [MB], $\omega_{\tilde{\mathcal{S}}}$ is ample. As ε is finite, $\varepsilon^*\omega_{\tilde{\mathcal{S}}} = \omega_{\mathcal{S}}$ is ample. \square

4.3 Arithmetic compactifications

Madapusi Pera has constructed in [Per] arithmetic toroidal compactifications $\mathcal{S}^{\Sigma} = \mathcal{S}_K^{\Sigma}(\mathbf{G}, \mathbf{X})$ (depending on a complete admissible rational partial polyhedral cone decomposition Σ for $(\mathbf{G}, \mathbf{X}, K)$) and the arithmetic minimal compactification $\mathcal{S}^{\min} = \mathcal{S}_K^{\min}(\mathbf{G}, \mathbf{X})$. Here we use the following facts about these compactifications.

- (1) For every Σ as above there exists a flat proper integral model over $O_{E,(v)}$ of the toroidal compactification of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ given by Σ which contains \mathcal{S} as an open dense subscheme ([Per] 4.6.13). It carries a canonical extension $\omega(\iota)^{\Sigma}$ of the Hodge line bundle ([Per] 4.8.1).
- (2) \mathcal{S}^{\min} is a projective $O_{E,(v)}$ -scheme containing \mathcal{S} as an open dense subscheme (by [Per] 4.8.11). It is flat with geometric normal fibers over $O_{E,(v)}$ (in the PEL case this is [Lan] 7.2.4.3 via the description of the local ring; for Shimura varieties of Hodge type the argument is the same).
- (3) For every Σ as above there is a natural proper surjective map $\int^{\Sigma}: \mathcal{S}^{\Sigma} \rightarrow \mathcal{S}^{\min}$ with geometrically connected fibers inducing the identity on \mathcal{S} ([Per] 4.8.11 (3)) such that $(\int^{\Sigma})^*\omega(\iota)^{\min} \cong \omega(\iota)^{\min}$.
- (4) For every p -integral Hodge embedding ι the Hodge line bundle $\omega_{\mathcal{S}}(\iota)$ extends to an ample line bundle $\omega_{\mathcal{S}}(\iota)^{\min}$ over \mathcal{S}^{\min} . Given different p -integral embeddings ι and ι' , there exists $s, s' \geq 1$ such that $(\omega_{\mathcal{S}}(\iota)^{\min})^{\otimes s'} \cong (\omega_{\mathcal{S}}(\iota')^{\min})^{\otimes s}$ ([Per] 4.8.11 (2)).
- (5) If $\mathrm{PGL}_{2,\mathbb{Q}}$ does not occur as a simple factor of \mathbf{G}^{ad} , then for any $k \geq 1$ pullback of sections induces isomorphism

$$(4.4) \quad \Gamma(\mathcal{S}^{\min}, (\omega(\iota)^{\min})^{\otimes k}) \xrightarrow{\sim} \Gamma(\mathcal{S}^{\Sigma}, (\omega(\iota)^{\Sigma})^{\otimes k}) \xrightarrow{\sim} \Gamma(\mathcal{S}, \omega(\iota)^{\otimes k}).$$

Indeed, the hypothesis implies that the codimension of $\mathcal{S}^{\min} \setminus \mathcal{S}$ is of codimension at least 2 in \mathcal{S}^{\min} . Then the second bijection is the Koecher principle ([Per] 4.8.12).

The composition is given by the pullback of the inclusion $\mathcal{S} \hookrightarrow \mathcal{S}^{\min}$. As \mathcal{S}^{\min} is normal, Hartogs' theorem implies that this composition is an isomorphism.

The same argument holds for the special fiber and the restriction of $\omega(\iota)^{\min}$ to the special fiber.

4.4 Ekedahl-Oort stratification

The conjugacy class $[\mu^{-1}]$ is defined over κ . As G is quasi-split, there exists a cocharacter χ of G_κ whose conjugacy class is $[\mu^{-1}]$. Then χ yields an orbitally finite algebraic zip datum as follows. Let $P_\pm = P_\pm(\chi)$ be the attached pair of opposite parabolic subgroups of G_κ with common Levi subgroup L the centralizer of χ . Let U_\pm be the unipotent radical of P_\pm . We obtain an algebraic zip datum $\mathcal{Z}_{G,\chi} := (G, P_+, P_-^\sigma, \varphi)$, where $(\)^\sigma$ denotes the pullback under absolute Frobenius $\sigma: x \mapsto x^p$ and where $\varphi: L \rightarrow L^\sigma$ is the relative Frobenius. In particular we obtain an attached zip group $E := E_{G,\chi}$. We set $P := P_+$ and $Q := P_-^\sigma$. Let U (resp. V) be the unipotent radical of P (resp. of Q). We define the algebraic quotient stack over κ

$$(4.5) \quad G\text{-Zip}^\chi := [E \backslash G_\kappa].$$

By [PWZ2] Proposition 3.11, for a κ -scheme T the T -valued points of this quotient stack is the groupoid of tuples $\underline{I} = (I, I_+, I_-, \iota)$, where I is a G_κ -torsor, where $I_+ \subseteq I$ a P -torsor, $I_- \subseteq I$ a Q -torsor, and where $\iota: I_+^\sigma/U^\sigma \xrightarrow{\sim} I_-/V$ is an isomorphism of L^σ -torsors.

The following lemma shows that we can assume (after conjugating χ over κ) that there exists a Borel pair (T, B) of G (defined over \mathbb{F}_p) such that χ is a B_κ -antidominant cocharacter of T_κ .

Lemma 4.2. *Let T be a maximal torus, B be a Borel subgroup of G containing T , and let B^- be the opposite Borel subgroup of B with respect to T (all defined over \mathbb{F}_p). Let $\chi: \mathbb{G}_{m,\kappa} \rightarrow G_\kappa$ be a cocharacter. Then there exists $g \in G(\kappa)$ such that for $\chi' := \text{int}(g) \circ \chi$ the following properties hold.*

- (1) $B_k \subseteq P_-(\chi')$.
- (2) χ' factors through T_κ .
- (3) $L(\chi')$ is the unique Levi subgroup of $P_-(\chi')$ containing T_κ . In particular $B^- \subseteq P_+(\chi')$.

The proof of the lemma works for all quasi-split reductive groups over arbitrary base fields.

Proof. By [SGA3] Exp. XXVI, Lemme 3.8 there exists a parabolic subgroup P' of G_k such that $B_k \subseteq P'$ and such that P' has the same type as $P_-(\chi)$. By loc. cit. Corollaire 5.5 (ii) there exists $g \in G(k)$ such that $P_-(\text{int}(g) \circ \chi) = {}^g P_-(\chi) = P'$. Hence we may assume that $B \subseteq P_-(\chi)$. Let L' be the unique Levi subgroup of $P_-(\chi)$ containing T . By loc. cit. Corollaire 1.8 there exists (a unique) $g \in U_-(\chi)(k)$ such that ${}^g L(\chi) = L'$. Hence we may assume that $L(\chi)$ is the Levi subgroup $P_-(\chi)$ containing T_k . But then T_k is a maximal torus of $L(\chi)$ and hence contains the identity component of the center of $L(\chi)$. By definition of $L(\chi)$, the cocharacter χ factors through the center of $L(\chi)$ and hence through its identity component. \square

From now on we choose T , B , and χ as in Lemma 4.2.

Zhang has constructed in [Zha1] a G -zip of type χ over $S_K := S_K(\mathbf{G}, \mathbf{X})$ and he has shown in loc. cit. that the corresponding classifying morphism $S_K \rightarrow G\text{-Zip}^\chi$ is smooth. Here we use the (slightly different) Construction 5.13 of a G -zip \underline{I} of type χ given by Wortmann in [Wor] §5 and obtain a smooth morphism

$$(4.6) \quad \zeta := \zeta_G: S_K \longrightarrow G\text{-Zip}^\chi.$$

The Ekedahl-Oort strata of S_K are the fibers of ζ .

Let us recall Wortmann's construction. Let $\bar{\mathcal{A}} \rightarrow S_K$ be the restriction of the universal abelian scheme to the special fiber. Define $\bar{\mathcal{V}} := H_{\text{DR}}^1(\bar{\mathcal{A}}/S_K)$, let $\mathcal{C} \subseteq \bar{\mathcal{V}}$ be the Hodge filtration, and let $\mathcal{D} \subseteq \bar{\mathcal{V}}$ be the conjugate filtration. The embedding ι (4.1) yields an embedding

$$(4.7) \quad G \hookrightarrow \text{GL}(\Lambda_{\mathbb{F}_p}) \xrightarrow{\sim} \text{GL}(\Lambda_{\mathbb{F}_p}^*),$$

where $(\)^*$ denotes the dual space and where the isomorphism is given by $g \mapsto g^\vee := {}^t g^{-1}$. Then $(\)^\vee \circ \chi$ and $(\)^\vee \circ \chi^\sigma$ define \mathbb{Z} -gradings

$$\Lambda_\kappa^* = \bigoplus (\Lambda_\kappa^*)_\chi^n, \quad \Lambda_\kappa^* = \bigoplus (\Lambda_\kappa^*)_{\chi^\sigma}^n$$

with $(\Lambda_\kappa^*)_\chi^n = (\Lambda_\kappa^*)_{\chi^\sigma}^n = 0$ for all $n \neq 0, 1$. We obtain a descending resp. ascending filtration

$$\begin{aligned} \text{Fil}_\chi^0 &:= \Lambda_\kappa^* \supset \text{Fil}_\chi^1 := (\Lambda_\kappa^*)_\chi^1 \supset \text{Fil}_\chi^2 := 0, \\ \text{Fil}_{-1}^{\chi^\sigma} &:= 0 \subset \text{Fil}_0^{\chi^\sigma} := (\Lambda_\kappa^*)_{\chi^\sigma}^0 \subset \text{Fil}_1^{\chi^\sigma} := \Lambda_\kappa^*. \end{aligned}$$

Then P_+ is the stabilizer of Fil_χ^\bullet in G_κ , and P_-^σ is the stabilizer of $\text{Fil}_\bullet^{\chi^\sigma}$ in G_κ .

Let $\bar{\mathcal{V}}_S = H_{\text{DR}}^1(\bar{\mathcal{A}}/S_K)$ be the restriction of \mathcal{V}_S to the special fiber. Let $\bar{s}_{\text{dR}} \subset \bar{\mathcal{V}}_S^\otimes$ be the reduction of the tensors $s_{\text{dR}} \subset \mathcal{V}_S^\otimes$, and denote by \bar{s} the base change of $s \in (\Lambda^*)^\otimes$ to $(\Lambda_\kappa^*)^\otimes$. Define

$$\begin{aligned} I &:= \mathcal{I}som_{S_K}((\Lambda_\kappa^*, \bar{s}) \otimes \mathcal{O}_{S_K}, (\bar{\mathcal{V}}_S, \bar{s}_{\text{dR}})), \\ I_+ &:= \mathcal{I}som_{S_K}((\Lambda_\kappa^*, \bar{s}, \text{Fil}_\chi^\bullet) \otimes \mathcal{O}_{S_K}, (\bar{\mathcal{V}}_S, \bar{s}_{\text{dR}}, \bar{\mathcal{V}}_S \supset \mathcal{C})), \\ I_- &:= \mathcal{I}som_{S_K}((\Lambda_\kappa^*, \bar{s}, \text{Fil}_\bullet^{\chi^\sigma}) \otimes \mathcal{O}_{S_K}, (\bar{\mathcal{V}}_S, \bar{s}_{\text{dR}}, \mathcal{D} \subset \bar{\mathcal{V}}_S)). \end{aligned}$$

Then G_κ acts from the right on I by $\beta \cdot g := \beta \circ g^\vee$, inducing right actions of P_+ and P_-^σ on I_+ and I_- , respectively. The Cartier isomorphism on $\bar{\mathcal{V}}_S$ induces an isomorphism $\iota: I_+^\sigma/U_+^\sigma \xrightarrow{\sim} I_-/U_-^\sigma$ and $\underline{I} := (I, I_+, I_-, \iota)$ is a G -zip of type χ over S_K ([Wor] 5.14). We obtain the morphism $\zeta: S_K \longrightarrow G\text{-Zip}^\chi$ (4.6), which is smooth by [Zha1] 3.1.2.

Remark 4.3. The following properties of the Ekedahl-Oort strata are known.

- (1) Each Ekedahl-Oort stratum is smooth ([WdYa]).
- (2) If the Ekedahl-Oort S^w is non-empty, it has dimension $\ell(w)$ ([Zha1] Proposition 3.1.6).

- (3) The closure of the Ekedahl-Oort stratum S^w is $\bigcup_{w' \preceq w} S^{w'}$ for a certain refinement \preceq of the Bruhat order; see [PWZ1] Definition 6.1 for the precise definition of \preceq ([Zha1] Proposition 3.1.6).
- (4) The inclusion $S^w \hookrightarrow \overline{S^w}$ is affine, in particular every irreducible component of $\overline{S^w} \setminus S^w$ is of codimension 1 in $\overline{S^w}$ ([WdYa]).

Remark 4.4. For Shimura varieties of PEL type it is shown in [ViWd] Theorem 10.1 that all Ekedahl-Oort strata are non-empty. In general this is expected, but it is not known.

Example 4.5. Consider the case $\mathcal{G} = \mathrm{GSp}(\Lambda)$. Recall that $\mathrm{rk}_{\mathbb{Z}(p)}(\Lambda) = 2g$. We endow Λ^* with the symplectic pairing ψ^* corresponding to the symplectic pairing ψ on Λ (i.e., if ψ is given by an isomorphism $\Lambda \xrightarrow{\sim} \Lambda^*$, then ψ^* is given by its inverse $\Lambda^* \xrightarrow{\sim} \Lambda = \Lambda^{**}$). Then $\mathrm{GL}(\Lambda) \xrightarrow{\sim} \mathrm{GL}(\Lambda^*)$, $g \mapsto g^\vee$ induces an isomorphism $\mathrm{GSp}(\Lambda) \xrightarrow{\sim} \mathrm{GSp}(\Lambda^*)$.

We have $\kappa = \mathbb{F}_p$ and $(\)^\vee \circ \chi = (\)^\vee \circ \chi^\sigma$ defines a decomposition $\Lambda_{\mathbb{F}_p}^* = (\Lambda_{\mathbb{F}_p}^*)_0 \oplus (\Lambda_{\mathbb{F}_p}^*)_1$ in totally isotropic subspaces. Hence P_+ and $P_- = P_-^\sigma$ are opposite parabolic subgroups whose common Levi subgroup is the stabilizer of the grading of $\Lambda_{\mathbb{F}_p}^*$.

By [PWZ2] 8.4, a $\mathrm{GSp}(\Lambda_{\mathbb{F}_p}^*)$ -zip of type $(\)^\vee \circ \chi$ (resp. a $\mathrm{GSp}(\Lambda_{\mathbb{F}_p})$ -zip of type χ) over an \mathbb{F}_p -scheme S may be interpreted as a triple $(\underline{\mathcal{M}}, \underline{\mathcal{L}}, E)$ consisting of an F -zip $\underline{\mathcal{M}}$ over S of rank $2g$ of type τ^\vee with $\tau^\vee(i) = g$ for $i = 0, 1$ (resp. of type τ with $\tau(i) = g$ for $i = -1, 0$), of an F -zip $\underline{\mathcal{L}}$ of S of rank 1 and an admissible epimorphism $E: \bigwedge^2 \underline{\mathcal{M}} \rightarrow \underline{\mathcal{L}}$ such that corresponding morphism of F -zips $\tilde{E}: \underline{\mathcal{M}} \rightarrow \underline{\mathcal{M}}^\vee \otimes \underline{\mathcal{L}}$ is an isomorphism. Here $\underline{\mathcal{M}}^\vee$ denotes the dual F -zip (in the sense of [PWZ2] Definition 6.5) and the tensor product is in the tensor category of F -zips over S ([PWZ2] Definition 6.4).

We apply functoriality of G -zips ([Zha2] Theorem 0.2) to the isomorphism $\mathrm{GSp}(\Lambda) \xrightarrow{\sim} \mathrm{GSp}(\Lambda^*)$. Then the correspondence in [PWZ2] 8.4 shows that there is an equivalence between $\mathrm{GSp}(\Lambda_{\mathbb{F}_p})$ -zips of type χ and $\mathrm{GSp}(\Lambda_{\mathbb{F}_p}^*)$ -zip of type $(\)^\vee \circ \chi$ which is given by attaching to a $\mathrm{GSp}(\Lambda_{\mathbb{F}_p})$ -zip $(\underline{\mathcal{M}}, \underline{\mathcal{L}}, E)$ of type χ the $\mathrm{GSp}(\Lambda_{\mathbb{F}_p}^*)$ -zip of type $(\)^\vee \circ \chi$ given by $(\underline{\mathcal{M}}^\vee, \underline{\mathcal{L}}^\vee, E^{-1})$. Here $E^{-1}: \bigwedge^2(\underline{\mathcal{M}}^\vee) \rightarrow \underline{\mathcal{L}}^\vee$ is the symplectic pairing given by

$$\tilde{E}^{-1} \otimes \mathrm{id}_{\underline{\mathcal{L}}^\vee}: \underline{\mathcal{M}}^\vee = \underline{\mathcal{M}}^\vee \otimes \underline{\mathcal{L}} \otimes \underline{\mathcal{L}}^\vee \xrightarrow{\sim} \underline{\mathcal{M}} \otimes \underline{\mathcal{L}}^\vee = (\underline{\mathcal{M}}^\vee)^\vee \otimes \underline{\mathcal{L}}^\vee.$$

Let $\mathcal{V}_{\tilde{S}}$ be the first De Rham cohomology of the universal abelian scheme over the special fiber $\tilde{S} := S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm)$. The Hodge filtration, the conjugate filtration, and the Cartier isomorphism define a natural structure of an F -zip $\mathcal{V}_{\tilde{S}}$ on $\mathcal{V}_{\tilde{S}}$ ([MoWd] §7). The universal equivalence class λ of prime to p quasi-isogenies yields a class of symplectic pairings on $\mathcal{V}_{\tilde{S}}$ up to multiplication with a locally constant function with values in \mathbb{F}_p^\times . We choose a pairing γ in this class. This yields by loc. cit. a morphism $\gamma: \bigwedge^2(\mathcal{V}_{\tilde{S}}) \rightarrow \mathbb{1}(1)$ of F -zips, where $\mathbb{1}(1)$ denotes the Tate- F -zip of weight 1 ([PWZ2] Example 6.6). One obtains a $\mathrm{GSp}(\Lambda_{\mathbb{F}_p}^*)$ -zip of type $(\)^\vee \circ \chi$. Its dual $(\mathcal{V}_{\tilde{S}}^\vee, \mathbb{1}(-1), \gamma^{-1})$ as explained above is then the $\mathrm{GSp}(\Lambda_{\mathbb{F}_p})$ -zips of type χ defining the morphism

$$\zeta: \tilde{S} \longrightarrow \mathrm{GSp}(\Lambda_{\mathbb{F}_p}) - \mathrm{Zip}^\chi.$$

Proposition 4.6. Assume that all Ekedahl-Oort strata are non-empty (e.g., if the Shimura variety is of PEL type, cf. Remark 4.4). Then the free rank of $\mathrm{Pic}(G\text{-Zip}^\chi)$ is the number of Ekedahl-Oort strata of codimension one.

Proof. The rank of $\text{Pic}(G\text{-Zip}^\chi)$ is the number of irreducible components of $G - U$, where U is the open E -orbit in G (Corollary 2.9). \square

Remark and Definition 4.7. As ζ is open, the preimage of the generic point of $G\text{-Zip}^\chi$ is open and dense in $S_K(\mathbf{G}, \mathbf{X})$. Moreover, Wortmann has shown ([Wor] Theorem 6.10) that this generic Ekedahl-Oort stratum is the μ -ordinary stratum. We denote it by $S^{\mu\text{-ord}} = S_K^{\mu\text{-ord}}(\mathbf{G}, \mathbf{X})$.

4.5 Bruhat stratification

We continue to assume P and Q contain both a Borel subgroup of G_κ which is already defined over \mathbb{F}_p . We call

$$\mathcal{B}_G^\chi := [P \backslash G_\kappa / Q]$$

the Bruhat stack attached to (G, χ) . Let

$$(4.8) \quad \beta_G: G\text{-Zip}^\chi \longrightarrow \mathcal{B}_G^\chi$$

be the canonical morphism.

Example 4.8. We set $\bar{\Lambda} := \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ endowed with the induced symplectic form and write by $2g = \dim_{\mathbb{F}_p}(\bar{\Lambda})$. Then $\tilde{\chi} := \iota \circ \chi$ is $\text{GSp}(\bar{\Lambda})(\kappa)$ -conjugate to a cocharacter of $\text{GSp}(\bar{\Lambda})_\kappa$ which is defined over \mathbb{F}_p . As explained in Example 4.5, $\tilde{\chi}$ defines a cocharacter of $\text{GSp}(\bar{\Lambda})_\kappa$ which has only weights -1 and 0 on $\bar{\Lambda}_\kappa$ and such that the weight decomposition $\bar{\Lambda}_\kappa = \bar{\Lambda}_{-1} \oplus \bar{\Lambda}_0$ is a decomposition into totally isotropic subspaces.

For every κ -scheme, $\mathcal{B}(\text{GSp}(\bar{\Lambda}), \tilde{\chi})(S)$ can be identified with the groupoid of triples $(\mathcal{M}, \mathcal{L}, E, \mathcal{C}, \mathcal{D})$, where \mathcal{M} is a finite locally free \mathcal{O}_S -module of rank $2g$, \mathcal{L} is a finite locally free \mathcal{O}_S -module of rank 1, $E: \bigwedge^2(\mathcal{M}) \rightarrow \mathcal{L}$ is an alternating pairing such that the corresponding homomorphism $\tilde{E}: \mathcal{M} \xrightarrow{\sim} \mathcal{M}^\vee \otimes \mathcal{L}$ is an isomorphism, and where $\mathcal{C}, \mathcal{D} \subseteq \mathcal{M}$ are Lagrangian submodules (i.e., \tilde{E} induces isomorphisms $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^\perp \otimes \mathcal{L}$ and $\mathcal{D} \xrightarrow{\sim} \mathcal{D}^\perp \otimes \mathcal{L}$).

Using [Wd2] Example 2.11 one sees that the map

$$\beta_{\text{GSp}(\bar{\Lambda})}: \text{GSp}(\Lambda_{\mathbb{F}_p})\text{-Zip}^{\tilde{\chi}} \longrightarrow \mathcal{B}(\text{GSp}(\bar{\Lambda}), \tilde{\chi})$$

is given by attaching to a $\text{GSp}(\bar{\Lambda})$ -zip $(\underline{\mathcal{M}}, \underline{\mathcal{L}}, E)$ of type χ (in the sense of 4.5) the tuple $(\mathcal{M}, \mathcal{L}, E, \mathcal{C}, \mathcal{D})$, where \mathcal{M} and \mathcal{L} is the underlying \mathcal{O}_S -module of $\underline{\mathcal{M}}$ and $\underline{\mathcal{L}}$ and where $\mathcal{C} = C^0(\underline{\mathcal{M}})$ and $\mathcal{D} = D_{-1}(\underline{\mathcal{M}})$.

Hence by Example 4.5 the composition

$$\beta_{\text{GSp}(\bar{\Lambda})} \circ \zeta_{\text{GSp}(\Lambda)}: \tilde{S} := S_{\tilde{K}}(\text{GSp}(\Lambda), S^\pm) \rightarrow \mathcal{B}_{\text{GSp}(\bar{\Lambda})}^{\iota \circ \chi}$$

is given by the tuple $(\mathcal{V}_{\tilde{S}}^\vee, \mathbb{1}(-1), \gamma^{-1}, \mathcal{C}, \mathcal{D})$, where \mathcal{C} (resp. \mathcal{D}) is the orthogonal complement of the Hodge filtration (resp. the conjugate filtration) in $\mathcal{V}_{\tilde{S}}$.

We have a diagram of morphisms of algebraic stacks over κ

$$(4.9) \quad \begin{array}{ccccc} S_K(\mathbf{G}, \mathbf{X}) & \xrightarrow{\zeta_G} & G\text{-Zip}^\chi & \xrightarrow{\beta_G} & \mathcal{B}(G, \chi) \\ \varepsilon \downarrow & & \downarrow \iota_Z & & \downarrow \iota_{\mathcal{B}} \\ S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm) & \xrightarrow{\zeta_{\mathrm{GSp}(\Lambda)}} & \mathrm{GSp}(\bar{\Lambda})\text{-Zip}^{\iota \circ \chi} & \xrightarrow{\beta_{\mathrm{GSp}(\bar{\Lambda})}} & \mathcal{B}(\mathrm{GSp}(\bar{\Lambda}), \iota \circ \chi) \end{array}$$

where the middle and the right vertical arrows are induced by functoriality by the embedding $\iota: G \rightarrow \mathrm{GSp}(\bar{\Lambda})$.

Lemma 4.9. *This diagram is commutative.*

Proof. This is clear for the right square and is a special case of Theorem 0.2 in [Zha2] for the left square. \square

4.6 Hasse invariants for the μ -ordinary locus

Let $(\mathcal{M}, \mathcal{L}, E, \mathcal{C}, \mathcal{D})$ be the universal tuple over $\mathcal{B}_{\mathrm{GSp}(\bar{\Lambda})}^{\iota \circ \chi}$ (Example 4.8). Let

$$\omega_{\mathrm{GSp}(\bar{\Lambda})}^{\flat} := \det(\mathcal{C}) \in \mathrm{Pic}(\mathcal{B}_{\mathrm{GSp}(\bar{\Lambda})}^{\iota \circ \chi}).$$

Proposition 4.10. *The pullback of $\omega_{\mathrm{GSp}(\bar{\Lambda})}^{\flat}$ under $\delta := \beta_{\mathrm{GSp}(\bar{\Lambda})} \circ \zeta_{\mathrm{GSp}(\Lambda)}$ is isomorphic to the Hodge line bundle on $S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm)$.*

Proof. Example 4.8 shows that $\delta^* \mathcal{M} \cong \mathcal{V}_{\tilde{S}}^\vee$. Moreover, E induces an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^\vee \otimes \mathcal{L}$ and hence $\det(\mathcal{M}^\vee) \cong \det(\mathcal{M}) \otimes \mathcal{L}^{\otimes -1}$. Moreover $\delta^*(\mathcal{C}) = (\mathcal{V}_{\tilde{S}}/\mathcal{H})^\vee$, where $\mathcal{H} \subset \mathcal{V}_{\tilde{S}}$ is the Hodge filtration. Hence the Hodge line bundle is given by

$$\det(\mathcal{H}) \cong \det(\mathcal{V}_{\tilde{S}}) \otimes \det((\mathcal{V}/\mathcal{H})^\vee) \cong \delta^*(\det(\mathcal{M}) \otimes \mathcal{L}^{\otimes -1} \otimes \det(\mathcal{C})).$$

Now $\det(\mathcal{M})$ is the line bundle attached to the determinant character of $\mathrm{GSp}(\Lambda)$, and \mathcal{L} is the line bundle attached to the multiplier character of $\mathrm{GSp}(\Lambda)$. Hence $\det(\mathcal{M}) \cong \mathcal{L}^{\otimes g}$. Moreover $\delta^*(\mathcal{L})$ is the underlying line bundle of the Tate F -zip $\mathbb{1}(-1)$ and hence is trivial. Therefore $\delta^*(\det \mathcal{M})$ is also trivial. This proves the claim. \square

We define a line bundle on \mathcal{B}_G^χ by

$$(4.10) \quad \omega_G^{\flat} := \iota_{\mathcal{B}}^*(\omega_{\mathrm{GSp}(\bar{\Lambda})}^{\flat}).$$

As the universal abelian scheme \mathcal{A} over $\mathcal{S}_K(\mathbf{G}, \mathbf{X})$ is defined as the pull back of the universal abelian scheme over $\mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^\pm)$, the same holds for the Hodge line bundle of \mathcal{A} . Hence the commutativity of (4.9) shows that

$$(4.11) \quad (\beta_G \circ \zeta_G)^*(\omega_G^{\flat}) = \omega_{S|S_K(\mathbf{G}, \mathbf{X})},$$

i.e., the pull back of ω_G^{\flat} is the Hodge line bundle on $S_K(\mathbf{G}, \mathbf{X})$. We define a line bundle on $G\text{-Zip}^\chi$ by

$$\omega_G^{\flat} := \beta_G^*(\omega_G^{\flat}).$$

Then $\zeta_G^*(\omega_G^b)$ is the Hodge line bundle on $S_K(\mathbf{G}, \mathbf{X})$ by (4.11). By Proposition 1.18, we have

$$(4.12) \quad \dim \Gamma(G\text{-Zip}^\chi, (\omega_G^b)^{\otimes r}) \leq 1$$

for all $r \in \mathbb{Z}$.

Definition 4.11. Let S_0 be the reduced stabilizer of 1 (see Subsection 2.5). The exponent $N := N_{\mathbf{G}, \mathbf{X}}$ of its character group $X^*(S_0)$ is called the *Hasse number of (\mathbf{G}, \mathbf{X})* .

Theorem 4.12. Let $N = N_{\mathbf{G}, \mathbf{X}}$ be the Hasse number. Then for every integer $d \geq 1$ we have

$$\dim \Gamma(G\text{-Zip}^\chi, (\omega_G^b)^{\otimes Nd}) = 1.$$

For every non-zero section $H \in \Gamma(G\text{-Zip}^\chi, (\omega_G^b)^{\otimes Nd})$ the non-vanishing locus of $\zeta_G^*(H) \in \Gamma(S_K(\mathbf{G}, \mathbf{X}), (\omega_S)^{\otimes ND})$ is the μ -ordinary locus S^μ of $S_K(\mathbf{G}, \mathbf{X})$.

We call any section of the form $\zeta_G^*(H)$ as above a *Hasse invariant (of parallel weight Nd)*.

Proof. By (4.12) it suffices to consider the case $d = 1$. Let $\tilde{P} = P_+(\iota \circ \chi)$ be the parabolic of $\tilde{G} := \mathrm{GSp}(\bar{\Lambda})$ defined by $\iota \circ \chi$. The pullback of $\det(\mathcal{C})$ to \tilde{G}/\tilde{P} is anti-ample. It is defined by a character $\tilde{\lambda} \in X^*(\tilde{P})$ by Corollary 1.6 because $\mathrm{Pic}(\tilde{G}) = 1$ (Corollary 2.2). Hence its restriction \mathcal{R} to G/P is anti-ample and defined by a G -anti-ample character $\lambda \in X^*(P)$, the restriction of $\tilde{\lambda}$. The parabolic P is defined by a minuscule cocharacter and in particular by a small cocharacter. Hence we can apply Theorem 3.8 which shows that there exists a section H of $(\omega_G^b)^{\otimes N}$ such that $\mathrm{div}(H)$ is an element of $Z^1(G)^E$ with coefficients > 0 (Remark 3.7). In particular, its support is the complement of the generic point in $G\text{-Zip}^\chi$. As ζ_G is flat (even smooth) and dominant, the vanishing locus of $\zeta_G^*(H)$ is the complement of the μ -ordinary stratum in $S_K(\mathbf{G}, \mathbf{X})$. \square

Corollary 4.13. Assume that $S_K(\mathbf{G}, \mathbf{X})$ is projective, then the μ -ordinary stratum S^μ is affine.

Proof. This follows from Theorem 4.12 because ω_S is ample by Proposition 4.1. \square

Remark 4.14. Madapusi Pera has shown that $S_K(\mathbf{G}, \mathbf{X})$ is projective if and only if \mathbf{G}^{ad} is an anisotropic group over \mathbb{Q} ([Per] 4.4.7).

More generally, assume that \mathbf{G}^{ad} has no factor isomorphic to $\mathrm{PGL}_{2, \mathbb{Q}}$. Then every Hasse invariant H of weight k extends uniquely to a section in $\Gamma(\mathcal{S}^{\min} \otimes \kappa, (\omega_S^{\min})^{\otimes k})$. We call its non-vanishing locus the μ -ordinary locus of \mathcal{S}^{\min} and denote it by $\mathcal{S}^{\min, \mu}$.

Lemma 4.15. The μ -ordinary locus $\mathcal{S}^{\min, \mu}$ does not depend on the choice of the Hasse invariant.

Proof. Let H and H' are Hasse invariants of weight k and k' , respectively. We may assume that $k' = dk$ for some $d \geq 1$. Then $H^{\otimes d}$ and H' differ only by a non-zero scalar. \square

As ω_S^{\min} is ample and \mathcal{S}^{\min} is projective, we deduce again:

Corollary 4.16. The μ -ordinary locus $\mathcal{S}^{\min, \mu}$ of the minimal compactification is affine.

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